Matroid theory
Exercise Sheet 8
Date: 21 April 2026

Exercise 8.1. Let the edges of a complete graph be colored in such a way that there is no triangle whose three edges have three different colors. Then there exists a monochromatic spanning tree.

Exercise 8.2. Let $M = (S, r)$ be a matroid, and let $e \in S$ be a non-loop element. Prove that exactly one of the following two conditions holds:

- There exists a set $X \subseteq S - e$ such that the matroid $M \setminus X$ has two disjoint bases, neither of which contains e.
- There exists a set $X \subseteq S - e$ such that, in M/X, the set $S \setminus X$ can be partitioned into two independent sets.

Exercise 8.3. Let A and B be bases of the matroid $M = (S, r)$. Prove that for any partition $A = A_1 \cup \cdots \cup A_q$ there exists a partition $B = B_1 \cup \cdots \cup B_q$ such that $A - A_i + B_i$ is a basis for $i = 1, \ldots, q$.

Exercise 8.4. Let $M = (S, r)$ be a loopless matroid and let $J \subseteq S$ be a subset of at most k elements. Prove that if S can be partitioned into k independent sets, then S can be partitioned into k independent sets in such a way that each of them contains at most one element of J.

Exercise 8.5. Let $M_1 = (S, r_1)$ and $M_2 = (S, r_2)$ be matroids. Prove that

$$|S| + \min\{r_1(X) + r_2(X) - |X| : X \subseteq S\} = \max\{r_1(X) + r_2(S - X) : X \subseteq S\}.$$

Exercise 8.6. Let $M_1 = (S, \mathcal{B}_1)$ and $M_2 = (S, \mathcal{B}_2)$ be matroids and $F \subseteq S$. Give a polynomial-time algorithm (using independence oracles) to determine whether there exist $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ bases such that $B_1 \cap B_2 = F$.

Exercise 8.7. Let $G = (A, B; E)$ be a bipartite graph that has a perfect matching, and let $w : E \to \mathbb{R}$ be a weight function. Then the maximum total weight of a perfect matching is equal to the minimum total weight of a weighted vertex cover, that is,

$$\max\left\{\sum_{e \in M} w(e) : M \text{ is a perfect matching}\right\} = \min\left\{\sum_{a \in A} x_a + \sum_{b \in B} y_b : \ x_a + y_b \geq w(ab) \quad \text{for every } ab \in E \ \right\}.$$

Homework (submission deadline: April 28)

Exercise 8.8. Prove that if $M_1 = (S, \mathcal{B}_1)$ and $M_2 = (S, \mathcal{B}_2)$ are matroids such that the ground set S can be partitioned into two bases in both matroids, and $c\colon S \to \mathbb{R}$ is a weight function, then there exists a common basis $B \in \mathcal{B}_1 \cap \mathcal{B}_2$ with weight at least $c(S)/2$.

Challenging problem (submission deadline: May 12)

Exercise 8.9. Consider the following open problems.

- (Woodall's conjecture) Let $D = (V, A)$ be a directed graph. We call a subset of edges $A' \subseteq A$ a *dicut*, if there exists a partition $V = V_1 \cup V_2$ of the vertices with no edges directed from V_2 to V_1, and the set of edges pointing from V_1 to V_2 is exactly A'. A *dijoin* is a subset of edges that intersects every dicut. Is it true that the minimum number of edges in a dicut of D is always equal to the maximum number of disjoint dijoins?
- (Barnette's conjecture) Is it true that every bipartite, 3-regular, planar, 3-vertex-connected graph has a Hamiltonian cycle?

Show that these conjectures can be reformulated as three matroids having a common basis.

Research opportunity

Question 8.10. There are several open problems that can be formulated in terms of proving that three matroids have a common basis. In general, this problem is hard. However, examining these examples and finding the right extra conditions might give a special case in which this problem is tractable, and hopefully it resolves some open problems at once.