Matroid theory
Exercise Sheet 7
Date: 14 April 2026

Exercise 7.1. Let $M_1 = (S, \mathcal{B}_1), M_2 = (S, \mathcal{B}_2), \ldots, M_k = (S, \mathcal{B}_k)$ be matroids. Prove that there exist bases $B_i \in \mathcal{B}_i$ in such a way that every element appears in at least two bases if and only if

$$\sum_{i=1}^{k} r_i(X) \geq 2|X|$$

for every $X \subseteq S$.

Exercise 7.2. Let $G = (V, E)$ be a graph, $U \subseteq V$ and $g\colon U \to \mathbb{Z}_+$.

a) Show that in general it is hard to decide whether there exists a spanning tree of G with degree at most $g(u)$ for all $u \in U$.

b) Show that if U is a stable set of G, then there exists a polynomial-time algorithm for deciding whether there exists a spanning tree of G with degree at most $g(u)$ for all $u \in U$.

Exercise 7.3. Let $M = (S, r)$ be a loopless matroid and let $J \subseteq S$ be a subset of at most k elements. Prove that if there are k disjoint bases, then there are k disjoint bases in such a way that each of them contains at most one element of J and their union contains J.

Exercise 7.4. Let G be an undirected graph whose edges are colored. Give a polynomial-time algorithm (using independence oracles) to determine whether there exists a spanning tree consisting of edges with pairwise distinct colors.

Exercise 7.5. Recall the matroid union algorithm. Let $M_1 = (S, \mathcal{F}_1), M_2 = (S, \mathcal{F}_2), \ldots, M_k = (S, \mathcal{F}_k)$ be matroids on the same ground set S. The algorithm provides $F_1, F_2, \ldots, F_k, X \subseteq S$ such that $F_1, F_2, \ldots, F_k$ are pairwise disjoint, $F_i \in \mathcal{F}_i$, $S = F_1 \cup \ldots \cup F_k \cup X$ and $F_i \cap X$ spans X in M_i for all $1 \leq i \leq k$. These conditions serve as a certificate for the set $F_1 \cup F_2 \cup \ldots \cup F_k$ being optimal. Prove that regardless of how the algorithm runs, the resulting set X is always the same.

Homework (submission deadline: April 21)

Exercise 7.6. Prove that if $M_1 = (S, \mathcal{B}_1)$ and $M_2 = (S, \mathcal{B}_2)$ are matroids such that the ground set S can be partitioned into two bases in both matroids, then there exists a $B \in \mathcal{B}_1 \cap \mathcal{B}_2$ common basis.

Challenging problem (submission deadline: May 5)

Exercise 7.7. Let $M = (S, \mathcal{F})$ be a matroid whose ground set decomposes into two disjoint bases, and consider a coloring of S such that each color is used at most twice.

a) Show that S can be covered by three rainbow independent sets of M.

b) Show that S can be covered by $\lfloor \log |S| \rfloor + 1$ rainbow bases.

Research opportunity

Question 7.8. Is there a constant $C > 0$ such that for any matroid $M = (S, \mathcal{F})$ whose ground set decomposes into two disjoint bases, and any coloring of S such that each color is used at most twice, S can be covered by C rainbow bases?