Matroid theory
Matroid invitational competition
Submission deadline: April 14

We say that a matroid M is *k-coverable* if it can be covered by k independent sets. Given matroids $M = (S, \mathcal{I})$ and $N = (S, \mathcal{J})$, we say that N is a *reduction* of M if $\mathcal{J} \subseteq \mathcal{I}$; that is, every independent set of N is also independent in M. We denote that N is a reduction of M by $N \preceq M$. A *unitary partition matroid* is a matroid $N = (S, \mathcal{J})$ such that $\mathcal{J} = \{X \subseteq S : |X \cap S_i| \leq 1 \text{ for } i = 1, \dots, q\}$ for some partition $S = S_1 \cup \dots \cup S_q$.

Problem 1. Prove that for every loopless matroid M of rank r there exists a unitary partition matroid N of rank r with $N \preceq M$.

Problem 2. a) Let $M = (S, \mathcal{I})$ be a k-coverable graphic matroid. Prove that there exists a $(2k-1)$-coverable unitary partition matroid N with $N \preceq M$.
b) Prove that this bound is tight in the sense that $(2k-2)$-coverable unitary partition matroid does not necessarily exist.

Problem 3. Show that a connected matroid is uniquely determined by the set of circuits containing a given element x.

Problem 4. Let r, g, b be non-negative integers. Let Γ be a connected graph on $r + g + b + 1$ vertices. Each edge of Γ is coloured red, green, or blue. It turns out that Γ has

- a spanning tree in which exactly r of the edges are red,
- a spanning tree in which exactly g of the edges are green, and
- a spanning tree in which exactly b of the edges are blue.

Prove that Γ has a spanning tree in which exactly r of the edges are red, exactly g of the edges are green, and exactly b of the edges are blue.

Problem 5. Given an undirected graph $G = (V, E)$ and two distinct vertices $s, t \in V$. Two players, Cut and Join, alternately claim previously unclaimed edges of E, with Cut moving first. Join wins if the edges claimed by Join contain an s–t path; otherwise, Cut wins. Prove that Join has a winning strategy if and only if there exists a subgraph $G' = (V', E')$ with $s, t \in V'$ such that G' contains two edge-disjoint spanning trees.

Problem 6. Let A and B be bases of the matroid $M = (S, r)$. Prove that for any partition $A = A_1 \cup A_2$ there exists a partition $B = B_1 \cup B_2$ such that $A - A_i \cup B_i$ and $B - B_i \cup A_i$ are both bases for $i = 1, 2$.

Problem 7. Let $G = (S, T; E)$ be a bipartite graph, and $M_1 = (S, r_1)$ and $M_2 = (T, r_2)$ be matroids. We call a matching $F \subseteq E$ strongly independent if it covers independent sets both in M_1 and M_2. Prove that the maximum size of a strongly independent matching is equal to

$$\min\{r_1(X) + r_2(Y) \mid X \subseteq S, Y \subseteq T, X \cup Y \text{ covers every edge of } G\}.$$