Graph Theory **Lecture date: March 16, 2026**
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1 Introduction

Nowhere-zero flows were introduced by Tutte in relation to graph colorings and chromatic polynomials [6]. The concept of nowhere-zero flows (NZF for short) has connections to other graph theoretical problems such as cycle covers [3, 7] and Woodall's conjecture about dicuts and dijoins [1]. Furthermore, NZFs can also be generalized from graphs to regular matroids [5].

Definition 1. Let $G = (V, E)$ be a graph and k a positive integer. We call a pair (D, x) a nowhere-zero k-flow (k-NZF) on G if

- $D = (V, A)$ is an orientation of G,
- $x : A \to [1-k, k-1] \cap \mathbb{Z}$ is a circulation on D, that is,

$$\sum_{uv \in A} x(uv) - \sum_{vw \in A} x(vw) = 0 \ \forall v \in V,$$

- $x(uv) \neq 0 \ \forall uv \in A$.

We can replace $[1-k, k-1] \cap \mathbb{Z}$ by an abelian group Γ to get the definition of nowhere-zero Γ-flows (Γ-NZFs).

Observation 1. *The following are equivalent:*

1. $\exists D$ *orientation of* G *such that* G *admits a* k*-NZF with orientation* D*.*
2. $\forall D$ *orientation,* G *admits a* k*-NZF with orientation* D*.*
3. G *admits a* k*-NZF* (D, x)*, where* $x(e) > 0 \ \forall e \in A$*.*

Sometimes it is convenient to work with finite groups instead of $\mathbb{Z}$, therefore we state the following theorem.

Theorem 2 (Jaeger [2])**.** *Let* $G = (V, E)$ *be a graph and* Γ *any finite group with* $|\Gamma| = k$*. Then* G *admits a* Γ*-NZF if and only* G *admits a* k*-NZF.*

2 Existence of a k-NZF

In planar graphs there is a clear connection between map-colorings and nowhere-zero flows captured in the following statement.

Theorem 3. *Let* $G = (V, E)$ *be a 2-edge-connected planar graph. Then the faces of* G *can be colored with* k *colors if and only if* G *admits a nowhere-zero* k*-flow.*

By the four color theorem, we know that the faces of every loopless planar graph can be properly colored using 4 colors. Based on the analogy between map-colorings and nowhere-zero flows it makes sense to ask the question, whether we have such universal upper bound for nowhere-zero flows. Before we answer this question, we show a few special cases for k-NZFs for small values of k.

Proposition 1. *A graph admits a* 2*-NZF if and only if every vertex of* G *has even degree.*

Proposition 2. *Let* G *be a* 3*-regular graph. Then*

1. *G admits a 3-NZF if and only if G is bipartite.*
2. *G admits a 4-NZF if and only if $\chi'(G) = 3$.*

Proof. The first part is left as an exercise for the reader. For the second part, by Theorem 2, it suffices to prove the statement for $\mathbb{Z}_2 \times \mathbb{Z}_2$-NZF instead of 4-NZF. Finishing the proof is an easy exercise left for the reader. □

To answer our question about the universal upper bound for a k-NZF, we first state Tutte's conjecture.

Conjecture 1 (Tutte, 1954 [6])**.** *Every bridge-less graph admits a 5-NZF.*

The condition that G is bridge-less is clearly necessary since a graph that has a bridge cannot have a k-NZF for any k. The next observation shows that 5 is the lowest upper bound we can hope for.

Observation 2. *The Petersen graph does not admit a 4-NZF.*

Proof. The edges of the Petersen graph cannot be colored with 3 colors, thus by Proposition 2 it cannot admit a 4-NZF. □

Conjecture 2. *If a bridge-less graph does not contain a subdivided Petersen graph, then it admits a 4-NZF.*

The first universal upper bound was proven by Jaeger in 1979.

Theorem 4 (Jaeger [2])**.** *Every bridge-less graph admits an 8-NZF.*

In the next section we show Seymour's improvement on this result to 6, which is the best known upper bound.

3 The main result

Theorem 5 (Seymour [4])**.** *Every bridge-less graph admits a 6-NZF.*

First we need the following observations:

Proposition 3. *For $k > 2$, if $G = (V, E)$ is a bridge-less graph such that G does not admit a k-NZF and $|V| + |E|$ is minimum, then G is simple, 3-regular and 3-connected.*

Proposition 4. *If $k > 0$ and Φ is an integer-valued circulation in $G = (V, E)$, then there is a k-flow Φ' such that for every $e \in E$,*

$$\Phi(e) \equiv \Phi'(e) \ (k).$$

To prove the main theorem, we need to introduce a closure operator on subsets of the edges:

Definition 6. For $G = (V, E)$ and an integer $k > 0$, for a subset $X \subseteq E$ we define $\langle X \rangle_3$ to be the smallest set $Y \subseteq E$ with the following properties:

- $X \subseteq Y$,
- there is no cycle C such that $0 < |C - Y| \leq k$.

It can easily be seen that $\langle X \rangle_k$ is well-defined because if Y_1 and Y_2 satisfy the above properties, then so does $Y_1 \cap Y_2$. An important property of $\langle X \rangle_k$ is that it is a closure operator, that is,

$$X \subseteq \langle X \rangle_k, \qquad \langle \langle X \rangle_k \rangle_k = \langle X \rangle_k, \qquad X \subseteq Y \Rightarrow \langle X \rangle_k \subseteq \langle Y \rangle_k.$$

Lemma 7. *Let $G = (V, E)$, $k > 0$, let $X \subseteq E$ be such that $\langle X \rangle_{k-1} = E$. Then G admits a k-flow Φ with $E - X \subseteq supp(\Phi)$.*

Proof. We proceed by induction on $|E - X|$. If this is zero the result is trivial, so we assume it is not the case. Then $\langle X \rangle_{k-1} \neq X$ and so there is a circuit C with $0 < |C - X| \leq k - 1$. By the properties of $\langle \cdot \rangle_k$, $\langle XUC \rangle_{k-1} = E$, and so by induction there is a k-flow Φ_1 with $E - (XUC) \subseteq \text{supp}(\Phi_1)$. Take a circulation Φ_2 so that $\text{supp}(\Phi_2) = C$ and $\Phi_2(e) = 0\ \forall e \in E - C, \Phi_2(e) = \pm 1\ \forall e \in C$. Choose an integer ℓ such that $0 \leq \ell \leq k - 1$ and $\Phi_1(e) + \ell \Phi_2(e) \not\equiv 0\ (k)$ for each $e \in C - X$ (this is possible since $|C - X| \leq k - 1$). Put $\Phi = \Phi_1 + \ell \Phi_2$. Then for $e \in E - (XUC)$, $\Phi(e) = \Phi_1(e)$ and for $e \in C - X$, $\Phi(e) = \Phi_1(e) + \ell \Phi_2(e) \not\equiv 0\ (k)$ by choice of ℓ. Thus for each $e \in E - X$, $\Phi(e) \not\equiv 0\ (k)$. The result follows from Proposition 4. ☐

Lemma 8. *If $G = (V, E)$ is a 3-connected graph and $|V| \geq 3$, then there are vertex-disjoint circuits $C_1, \ldots, C_r$ such that $\langle C_1 \cup \ldots \cup C_r \rangle_2 = E$.*

A key idea of proving the lemma is to prove the following:

Lemma 9. *Let $G = (V, E)$ be a simple graph with $d(v) \geq 2$ for every vertex $v \in V$. Then there is a subgraph B of G with at least 3 vertices, such that B is 2-connected and at most one vertex of B is adjacent in G to vertices of G not in B.*

We do not show the proof of either lemma here, they can be found in [4].

Now we finally prove the main theorem using this lemma.

Proof of Theorem 5. By Proposition 3, it is enough to prove the theorem for 3-regular, simple, 3-connected graphs. In this case, by Lemma 8, there exits a subset $X \subseteq E$ such that $X = C_1 \cup \ldots \cup C_r$ for some vertex-disjoint circuits $C_1 \ldots C_r$ and $\langle X \rangle_2 = E$. By Lemma 7, there is a 3-flow Φ_1 such that $E - X \subseteq \text{supp}(\Phi_1)$. Let Φ_2 be a 2-flow on X. Then $\Phi = \Phi_1 + 3\Phi_2$ is clearly a flow. For an edge $e \in E - X$, $\Phi(e) = \Phi_1(e) \neq 0$ and $|\Phi(e)| \leq 2$. For an edge $e \in X$, $\Phi(e) = \Phi_1(e) \pm 3 \neq 0$ and $|\Phi(e)| \leq 5$, thus Φ is a nowhere-zero 6-flow. ☐

References

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