Graph Theory	**Lecture date: March 23, 2026**
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1 Introduction

Chordal graphs form a particularly well-behaved and highly structured class of graphs, where many problems that are computationally difficult in general become efficiently solvable. While tasks such as graph coloring, maximum clique, or maximum independent set are typically NP-hard, chordal graphs admit algorithms that solve these problems in polynomial—often even linear—time.

2 Perfect Elimination Ordering

Definition 1. An undirected graph $G = (V, E)$ is **chordal** if all cycle of size at least 4 admits a chord.

Definition 2. A vertex $v \in V$ is **simplicial** if forms a clique with its neighbors.

Definition 3. We call an ordering α of the vertices of G a **perfect elimination ordering** if for all vertex $v_i \in V$ its neighbors v_j, $j > i$ form a clique.

Theorem 4. *A graph G is chordal if and only if every minimal vertex-cut of G induces a clique.*

Lemma 5. *Every chordal graph G has a simplicial vertex. If G is not complete, then it has two nonadjacent simplicial vertices.*

Theorem 6. *A graph G is chordal if and only if G has a perfect elimination ordering.*

Proof. $(\rightarrow)$ Suppose G is chordal. Proof by induction on the number of vertices. For $n = 1$ this is trivial. Suppose $n > 1$ and every chordal graph with fewer vertices has a perfect elimination ordering. By the previous lemma G has a simplicial vertex v. The graph $G - \{v\}$ has a perfect elimination ordering $v_1, v_2, ..., v_n$. The ordering $v, v_1, ..., v_n$ is a perfect elimination ordering of G.

$(\leftarrow)$ Suppose G has a perfect elimination ordering. For an arbitrary cycle C consider the vertex $v \in V(C)$ with the smallest index in the ordering. Since it is a perfect elimination ordering the neighbors of v with greater indices form a complete graph. Thus, there is an edge (chord) between the two neighbors of v in C. □

3 Maximum Cardinality Search

A linear running-time algorithm for finding a perfect elimination ordering:

Algorithm 1 Maximum Cardinality Search

$\mathcal{L}_{n+1} \leftarrow \emptyset$
for $i = n...1$ $[-1]$ **do**
  choose vertex $v \in V - \mathcal{L}_{i+1}$ for which $|N(v) \cap V(\mathcal{L}_{i+1})|$ is maximal
  $\alpha(v) \leftarrow i$
  $\mathcal{L}_i \leftarrow \mathcal{L}_{i+1} \cup \{v\}$
end for

Lemma 7. *An ordering α of the vertices in a graph G is not a perfect elimination ordering if and only if for some vertex v, there exists a chordless path of length greater than one from $v = \alpha^{-1}(i)$ to some vertex in $\mathcal{L}_{i+l}$ through vertices in $V - \mathcal{L}_i$.*

Theorem 8. *Every maximum cardinality search ordering of a chordal graph G is a perfect elimination ordering.*

4 Algorithmic Applications

These problems are generally NP-hard, but for chordal graphs they are solvable in polynomial (linear) time:

- maximum clique $\omega(G)$
- maximum independent set $\alpha(G)$
- chromatic number $\chi(G)$

Definition 9. A graph G is perfect if $\omega(H) = \chi(H)$ for all induced subgraph $H \subseteq G$.

Theorem 10. *Chordal graphs are perfect.*

5 Clique trees

Let $G = (V, E)$ be a graph. Let $\mathcal{K}_G = \{K_1, ..., K_m\}$ denote the set of inclusion-wise maximal cliques of G.

Definition 11. We say that a tree $T = (\mathcal{K}_G, \mathcal{E})$ satisfies the **clique-intersection property** if for every pair of distinct cliques $K_i, K_j \in \mathcal{K}_G$ the set $K_i \cap K_j$ is contained in every clique on the path containing K_i and K_j on the tree.

Theorem 12. *A connected graph G is chordal if and only if there exists a tree $T = (\mathcal{K}_G, \mathcal{E})$ for which the clique-intersection property holds.*

Definition 13. We say that a tree $T = (\mathcal{K}_G, \mathcal{E})$ satisfies the **induced-subtree property** if for every vertex $v \in V$ the set $\mathcal{K}_G(v)$ induces a subtree of T.

Theorem 14. *For every connected graph G the set of trees that satisfy the clique-intersection property is equal to the set of trees that satisfy the induced-subtree property.*

Theorem 15. *A connected graph G is chordal if and only if there exists a tree $T = (\mathcal{K}_G, \mathcal{E})$ for which the induced-subtree property holds.*

References

[1] J. R. S. Blair and B. Peyton. An introduction to chordal graphs and clique trees. In A. George, J. R. Gilbert, and J. W. H. Liu, editors, *Graph Theory and Sparse Matrix Computation*, pages 1–29, New York, NY, 1993. Springer New York.

[2] A. Frank. Connections in combinatorial optimization. *Discrete Applied Mathematics*, 160:1875, 08 2012.