Graph Theory	**Lecture date March 30, 2026**
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1 Introduction

1.1 Definitions and basics

Let us introduce and refresh the basics of the planar graphs. [2]

Definition 1. A graph G is planar if it can be drawn in the plane such that no two edges intersect each other unless at a common vertex.

Definition 2. A plane graph is a drawing of a planar graph such that no two edges intersect each other unless at a common vertex.

Theorem 3 (Euler's formula)**.** *Let G be a connected plane graph with n vertices, m edges, and f faces. Then*

$$n - m + f = 2$$

2 Triangulations

Definition 4. A plane graph is a triangulation if it is simple, connected, and the degree of every face is 3.

Claim 5. *The triangulation of a graph is not unique.*

Claim 6. *A plane graph of order at least 3 (meaning it has at least 3 vertices) is maximally plane if and only if it is a plane triangulation.*

Corollary 7. *By theorem 3 and by claim 6 a plane graph with $n \geq 3$ vertices has at most $m = 3n-6$ edges. Every plane triangulation with n vertices has $3n-6$ edges. (If we do not assume that the exterior face is a triangle as well, then the bound would be $2n-6 \leq m \leq 3n-6$.)*

Corollary 8. *By the previous corollary we can see that in every maximal triangulation the average degree is:*

$$\frac{2m}{n} \leq \frac{6n-12}{n} = 6 - \frac{12}{n} < 6.$$

Hence we know that there must be a vertex of degree at most 5.

Claim 9. *Every maximal planar graph with at least four vertices is 3-connected.*

2.1 Theorems using triangulations

Theorem 10 (Fáry-Wagner)**.** *Every simple planar graph can be drawn so that all edges are straight line segments. [6]*

Proof. (sketch)
We are having the following lemmas for the proof:
1. If a graph is equal to a straight graph, then any of its subgraphs is also equal to a straight graph.
2. Every planar simple graph is a subgraph of a triangulated simple graph.
3. In a triangulated graph, the neighbors of a vertex Q are creating a cycle around Q which separates this vertex from the other parts of the graph.
4. If we delete Q from the cycle $C = P_1P_2....$, and we connect P_1 with each of the neighbors of Q to create a triangulated graph, we get a new graph G^*. If G^* is not a simple graph, then G had a circuit of three edges which separates two nodes in G.

5. Every triangulated simple graph is equal to a straight graph.

For the fifth lemma of the proof we are going to use induction on the number of vertices. We assume the statement holds for n vertices. Then by having a graph G with $n+1$ vertices and removing one vertex Q from G with the previously given method, we have G^* with n vertices.

If G^* is simple, then by induction hypothesis, we know that there exists a $G^{*'}$ straight-line graph with the corresponding cycle C'. If we take the neighbors in C' (ex. P_iP_j) and we take the halfplane which contains P_1 after each pair, then we get a convex region K'. Then every point of the intersection of K' and the interior of C', can be chosen as Q' and where the union of G^* and Q' will create a straight-line drawing, while still defining the same planar graph as G.

If G^* is not simple, then by the 4.lemma we have a triangle A in G that separates at least one vertex. Then, by induction hypothesis we can draw with straight lines what is inside A: $A-$ and what is outside A: $A+$. Then, with affine transformations, we can get the exterior triangle of $A-$ and the interior triangle of $A+$ to be the same size and glue the two graphs together while creating a straight-line drawing of the full graph G. □

Theorem 11 (Steinitz)**.** *A finite graph is 3-polytopal (or the 1-skeleton of a convex - 3D - polyhedron) if and only if it is planar and 3-connected. [4]*

Proof. (sketch)
→
We chose one face of the polyhedron, this chosen face becomes the outer face, so we can get a plane graph and of course we can not delete any two vertices of the polyhedron and get it disconnected because it would contradict that it is in 3D. By Balinski's theorem [1] we have a stronger statement too: the graph (1-skeleton) of a d-dimensional convex polytope is d-connected.
←
1. Reduce to triangulations
2. Realize triangulations as convex polyhedra
Let us use induction with local operators (ΔY- and $Y\Delta$-transformations): In the plane triangulation T, we know that every degree is at least 3, because of 3-connectedness and by corollary 8 we know that there is a vertex with degree at most 5. Let us delete this vertex v. Then the neighboring vertices form a cycle and we can triangulate this circle so the graph remains 3-connected (by reduction lemma) and we get T'. Let us call the according polyhedron Q'. Then we can add back vertex v to Q' with stacking on a triangular face, or using the visibility lemma in the cases the degree of v was 4 or 5. This completes our induction.
3. Remove the added edges

□

Theorem 12 (Circle Packing Theorem)**.** *[8] For every finite planar graph G, there is a circle packing in the plane with nerve G. If G is a plane triangulation the packing is unique.*

Theorem 13 (Four color theorem (v1))**.** *Every planar map can be colored with 4 colors so that adjacent regions have different colors.*

Using dual graphs, we have the following equivalent statement:

Theorem 14 (Four color theorem (v2))**.** *Every loopless planar graph is 4-colorable.*

The proof is too difficult to be part of this presentation, but here is a short historical background:
The first proof of the theorem that has not been disproved, came from Appel–Haken. Their essentially algorithmic proof uses triangulations (every plane triangulation must contain at least one of 1482 certain 'unavoidable configurations'):

Lemma 15. *The planar graph is 4-colorable if its related triangular graph is 4-colorable,* and they also used computer in their proof (to show that each of those configurations is ‘reducible’, meaning it cannot appear in a minimal counterexample). The basic logic was:

Any minimal counterexample must contain one of the configurations
$\downarrow$
All those configurations are reducible
$\downarrow$
No minimal counterexample can exist

Not only because of using the computer, but their proof have faced many criticism. N. Robertson, D. Sanders, P.D. Seymour and R. Thomas have since published a shorter proof using the same basics. [7]

2.2 Hamiltonicity

Definition 16. We define NST–triangulations to be those planar graphs that are triangulations in the sense that every face, perhaps except for the exterior face, is a triangle but that there is no separating triangle.

Theorem 17 (Whitney)**.** *Let G be a NST-triangulation. Then G is hamiltonian. [5]*

Tutte succeded to generalize Whitney’s result:

Theorem 18 (Tutte)**.** *Let G be a 4–connected planar graph. Then G is hamiltonian. [5]*

Corollary 19 (Thomassen)**.** *Every 4–connected planar graph has a hamiltonian cycle through any edge of G.*

Theorem 20 (Ewald)**.** *If G is a planar triangulation with $\Delta(G) \leq 6$ (where Δ is the largest degree), then G is hamiltonian. [5]*

3 Quadrangulations

Definition 21. A quadrangulation is a plane graph in which every face (including the outer face) is bounded by a cycle of length 4. In other words, every face is a quadrilateral.

Theorem 22. *Every quadrangulation is bipartite (2-colorable).*

Theorem 23. *The dual graph of a quadrangulation is 4-regular.*

3.1 Relations with triangulations

Theorem 24. *Every quadrangulation corresponds to a perfect matching in the dual of a triangulation.*

Corollary 25. *T admits a quadrangulation without Steiner points if and only if the dual graph G^* has a perfect matching.*

There are algorithms to create quadrangulations from triangulations in linear time, while adding at most $\lfloor \frac{n}{3} \rfloor$ outer Steiner points to the polygon.

4 Algorithms

Tarjan and van Wyk (1988) produced an $O(n \log \log n)$ algorithm for creating triangulations. This was followed by an unexpected result due to Chazelle (1991), who showed that an arbitrary simple polygon can be triangulated in $O(n)$. However ”this algorithm is quite hopeless to implement.”

4.1 Delaunay triangulation

Definition 26 (Voronoi cell)**.** Consider a finite set $S = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^2$ of n distinct points in the plane. The Voronoi cell V_i of $p_i \in S$ is the set of points x that are closer to p_i than to any other point of the set:

$$V_i = \{x \in \mathbb{R}^2 \mid \|x - p_i\| < \|x - p_j\|,\ \forall 1 \leq j \leq n,\ j \neq i\}.$$

Definition 27 (Delaunay triangulation)**.** The Delaunay triangulation $DT(S)$ is the geometric dual of the Voronoi diagram, it also satisfies the following properties:
1. There is no vertex inside the circumcircle of any triangle in the set.
2. In the plane, the Delaunay triangulation maximizes the minimum angle in 2D.
3. A Delaunay triangulation is unique (if no 4 point are cocircular).
4. The boundary of a Delaunay triangulation for a set of vertices V is the convex hull of V. [3]

Algorithm 1 Bowyer–Watson algorithm for Delaunay triangulation

Require: A finite set $S \subset \mathbb{R}^2$ of points
Ensure: The Delaunay triangulation of S
Construct a super-triangle containing all points of S
Let $\mathcal{T}$ be the triangulation consisting only of this super-triangle
for each point $p \in S$ **do**
    Let $\mathcal{B}$ be the set of all triangles in $\mathcal{T}$ whose circumcircle contains p
    Let $\partial\mathcal{B}$ be the set of boundary edges of the polygonal hole formed by removing $\mathcal{B}$
    Remove all triangles in $\mathcal{B}$ from $\mathcal{T}$
    for each edge e in $\partial\mathcal{B}$ **do**
        Add to $\mathcal{T}$ the triangle formed by p and e
    end for
end for
Remove all triangles that share a vertex with the super-triangle
return $\mathcal{T}$

References

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