

1 Classical properties and results on tournaments

A *tournament* is an orientation of a complete graph.

Theorem 1 (Rédei, 1934). *Every tournament contains a directed Hamiltonian path [6].*

Proof is trivial by induction.

Theorem 2. *A tournament is strongly connected if and only if it contains a directed Hamiltonian cycle.*

Proof. A Hamiltonian cycle is a clear evidence of strong connectivity. Now we show the other direction. Let C be the largest directed cycle. If it contains all vertices, we are done. Assume it does not. Now we take a vertex v outside C . If v dominates some vertex of C , then it also dominates the predecessor of v in C (otherwise we found a longer cycle). So then it dominates all vertices of C . Similarly if some vertex of C dominates v , then all vertices of C dominate v . Hence, all vertices outside of C are partitioned into two sets, one of which dominates C and the other is dominated by C . We must be able to reach the first set from the second, but then using the vertices of C together with 1 vertex from each set we easily get a larger cycle which gives a contradiction. \square

Some more general results with similar proofs:

Theorem 3. *Every strongly connected tournament is vertex-pancyclic, that is, every vertex lies on a directed cycle of every length from 3 to n [5].*

Theorem 4. *If a tournament T is a subgraph of a strongly connected digraph D , then there exists a directed cycle in D containing all vertices of T [5].*

Another standard extremal feature of tournaments is the existence of large transitive subtournaments.

Theorem 5. *Every tournament on n vertices contains a transitive subtournament on $1 + \lfloor \log_2 n \rfloor$ vertices [2].*

Proof sketch. We use induction on n , base case is trivial. Pick any vertex v , then one of $N^+(v), N^-(v)$ has size $\geq (n-1)/2$, WLOG, $N^+(v)$ has size $\geq (n-1)/2$. By induction it contains a transitive subtournament on $1 + \lfloor \log_2 |N^+(v)| \rfloor$ vertices; since v beats all of them, prepending v gives one of size $2 + \lfloor \log_2 |N^+(v)| \rfloor \geq 1 + \lfloor \log_2 n \rfloor$. \square

Theorem 6. *There exists a tournament on n vertices without a transitive subtournament of size $2 + 2 \log_2 n$ [2].*

(Fun fact: we can prove these kind of statements using Kolmogorov complexity.)

Also, in Ramsey theory, a central theme is that forbidding a fixed subtournament H often forces the existence of a large transitive subtournament [5].

Score sequences give another useful structural description. The *score* of a vertex is its outdegree.

Theorem 7 (Landau [4]). *A sequence $s_1 \leq s_2 \leq \dots \leq s_n$ of integers is the score sequence of a tournament if and only if*

$$\sum_{i=1}^k s_i \geq \binom{k}{2} \quad (1 \leq k \leq n),$$

with equality when $k = n$.

The proof follows easily from the well-known Orientation lemma.

Theorem 8 (Sands–Sauer–Woodrow, 1982 [7]). *Every 2-edge-colored tournament contains a vertex v such that every other vertex has a monochromatic directed path to v .*

The proof goes by a tricky induction on the number of vertices.

2 The absorbing/super vertex problem for more colors

While the first section was about some standard results, in this section we take Theorem 8 from the first section and explore some generalizations.

In the original form, the statement is false for more than 2 colors, as a simple triangle counterexample shows that. What extra condition can we add to make it true?

Theorem 9 (Shen, 1988 [8]). *Let T be an m -colored tournament. If no triangle of T is rainbow, then there is a vertex v such that every other vertex reaches v by a monochromatic directed path.*

Lemma 10 (Minimal counterexample lemma). *Let T be a minimal counterexample to the theorem above. Then T has a directed Hamiltonian cycle C such that every vertex of T monochromatically dominates every other vertex except its predecessor on C .*

Proof sketch. Let's choose a directed cycle C such that no vertex on C monochromatically dominates its predecessor. By minimality, C must be Hamiltonian. If some vertex u fails to dominate a non-predecessor v , then we can get a shorter directed cycle with the same property, which is a contradiction. \square

The proof of Theorem 9 uses the minimal counterexample lemma, but it will not be discussed here.

Now instead of avoiding some configurations, we can also ask for a dominating set of bounded size instead of a single vertex.

Theorem 11 (Erdős–Sands–Sauer–Woodrow, Bousquet–Lochet–Thomassé [1, 7]). *For every k there exists $f(k)$ such that every k -colored tournament contains a set S of size at most $f(k)$ from which every vertex is reachable by a monochromatic directed path.*

Now we build up a proof sketch of this result.

Definition 12. A *quasi-order* is a reflexive transitive relation. For each color i , define

$$x \leq_i y \iff \text{there is a monochromatic directed path of color } i \text{ from } x \text{ to } y.$$

Then each \leq_i is a quasi-order.

With this definition, we can reformulate Theorem 11 as follows: if the complete multidigraph on V is the union of quasi-orders P_1, \dots, P_k , then there is a set S of size at most $f(k)$ such that for every vertex $v \in V$ there is some $s \in S$ and some i such that $s \leq_i v$.

Definition 13. For a quasi-order P on A and $x \in A$, let

$$N_P^-(x) = \{y \in A : y \leq_P x\}.$$

A set $B \subseteq A$ is ε -dense in P if there exists a probability distribution w on A such that

$$w(N_P^-(x)) \geq \varepsilon \quad (x \in B).$$

Lemma 14 (Weighted partition lemma [1]). *Let the complete multidigraph on V be the union of quasi-orders P_1, \dots, P_k . Then there are a probability distribution w on V and a partition*

$$V = T_1 \cup \dots \cup T_k$$

such that for every $x \in T_i$,

$$w(N_{P_i}^-(x)) \geq \frac{1}{2k}.$$

Proof idea. Fisher–Ryan [3] gives $w(N^-(x)) \geq \frac{1}{2}$ for every vertex x . Since

$$N^-(x) = \bigcup_{i=1}^k N_{P_i}^-(x),$$

some i satisfies $w(N_{P_i}^-(x)) \geq \frac{1}{2k}$. □

Lemma 15 (BLT probabilistic lemma [1]). *For every $\varepsilon > 0$ there exists $g(\varepsilon)$ such that the following holds. If $C \subseteq B \subseteq A$, and both B and C are ε -dense in the same quasi-order P (first in A , then in B), then some set $S \subseteq A$ with $|S| \leq g(\varepsilon)$ dominates C .*

Proof sketch. Choose distributions w_A, w_B witnessing the two density conditions. Sample m vertices independently from A according to w_A , where

$$(1 - \varepsilon)^m < \varepsilon.$$

Then every $x \in B$ is missed with probability $< (1 - \varepsilon)^m < \varepsilon$, so for some sample-set S we have

$$w_B(N_P^+(S)) > 1 - \varepsilon.$$

Now every $y \in C$ satisfies $w_B(N_{P_i}^-(y)) \geq \varepsilon$, hence

$$N_P^+(S) \cap N_{P_i}^-(y) \neq \emptyset.$$

By transitivity, some $s \in S$ satisfies $s \leq_P y$, so S dominates C . □

The proof of the main result (Theorem 11) uses these two ingredients. We will discuss a sketch here.

We apply the weighted partition lemma recursively $k + 1$ times, each class is then indexed by a sequence $(j_1, \dots, j_{k+1}) \in [k]^{k+1}$. By the pigeonhole principle, some index repeats, e.g. $j_a = j_b$. Along that branch we can get two nested subsets that are $\frac{1}{2k}$ -dense in the same quasi-order, so we can apply the probabilistic lemma for them to get a bounded set dominating the final class. We can do the same for each final class and there are just k^{k+1} of them, so by taking the union of these dominating sets, we get a dominating set for which the size only depends on k which finishes the proof of Theorem 11[ESSW].

References

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