Graph Theory **Lecture date: March 16, 2026**

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1 Introduction

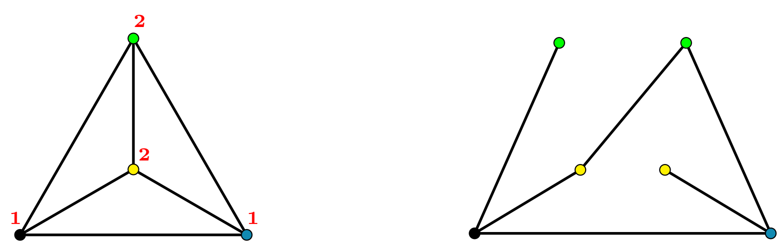
The detachment of a graph is when we split the vertices and then we distribute the edges among them. This technique was introduced by Nash-Williams, and it has many applications. First, we will see the main theorem in this topic, then other results related to the topic of connection.

2 Definitions and the theorem of Nash-Williams

Let $G = (V, E)$ be an undirected graph with $|V| = n$, $|E| = m$ and degree funtion d.

Definition 1. ([1], [2]) Let $r : V \to \mathbb{Z}^+$, where $r(v) \le d(v)$ for every $v \in V$. An r-detachment of G is a graph H obtained by "splitting" each vertex $v \in V$ into $r(v)$ vertices. The vertices $v_1, ..., v_{r(v)}$ obtained by splitting v are called the pieces of v in H. Every edge $uv \in E$ corresponds to an edge of H connecting some piece of u to some piece of v.
An r-degree specification is a function f on V, s.t. for every vertex $v \in V$, $f(v)$ is a sequence $d_1^v, ..., d_{r(v)}^v$ of positive integers s.t. $\sum_{i=1}^{r(v)} d_i^v = d(v)$. An f-detachment of G is an r-detachment in which the degrees of the pieces of each $v \in V$ are given by $f(v)$.

It is better if in an f-detachment the values of $f(v)$ are "*almost equal*", and the edges between u, v are distributed "*equally*".
For example: $d(v) = 7$ and $r(v) = 3$, then let the degrees be $2, 2, 3$.
Another example:



Theorem 2. *(**Nash-Williams**) [1] There exists a connected r-detachment iff for every $X \subseteq V$, $r(X) + c(X) \le e(X) + 1$, where $r(X) = \sum_{v \in X} r(v)$, $c(X)$ is the number of components of $G - X$ and $e(X)$ denotes the number of edges which have at least one endpoint in X. Furthermore, if G has a connected r-detachment, then G has a connected f-detachment for every r-degree specification f.*

Proof. (sketch) Let H be a connected r-detachment of G. We take an arbitrary $X \subseteq V$. In H we contract the pieces of the vertices of X and then contract the connected components of $G - X$. Then we get a H' graph, where $|V(H')| = r(X) + c(X)$ and $|E(H')| = e(X)$. H is connected, and the edges between the components of H' are still in H', so it is also connected. $|E(H')| \ge |V(H')| - 1 \Leftrightarrow e(X) \ge r(X) + c(X) - 1$.
In the other direction, we create a graph $G' = (V', E')$ obtained from G by replacing each vertex $v \in V$ by $r(v)$ vertices and by adding every uv edge to the graph $r(u)r(v)$ times (between every piece of u and every piece of v there is an uv edge). We color the edges with different colors, but every edge belongs to $uv \in E$ get the same color in G'. There is a connected r-detachment iff there is a rainbow spanning tree of G'. They are equivalent with this: There exist the basis of a graph matroid of G' which is also the independent set of a partition matroid of color assignment. Using matroid theorems we have that "$r(X) + c(X) \le e(X) + 1$" is necessary and sufficient condition to have a rainbow spanning tree in G'. □

There is another proof which uses orientations ([6]).
Also, there is a similar theorem (from Nash-Williams [3]) which gives necessary and sufficient conditions to have a k-edge-connected r-detachment.
There are other ways of using detachment, for example Hiroshi Nagamochi presented an application of connected detachments to molecular structure analysis ([5]).

3 Using of detachments

We say that graph G is *k-partition-connected* if it contains k edge-disjoint spanning trees. Now we show two theorems, from which the third follows.

Theorem 3. *[2] Let $G = (V, E)$ be a graph and $k \in \mathbb{N}^+$. Then G has a k-partition-connected r-detachment iff $i(X_0)+e(\mathcal{P}) \geq k\cdot(t-1)+k\cdot r(X_0)$ for all partitions $\mathcal{P} = \{X_0, X_1, ..., X_t\}$ of V, where X_0 can be empty or maybe $t = 0$. Furthermore, if G has a k-partition-connected r-detachment then G has a k-partition-connected f-detachment for every r-degree specification f in which $d_i^v \geq k$ for all $v \in V$ and $1 \leq i \leq r(v)$.*

Theorem 4. *[2] Let $G = (V, E)$ be a graph, let $r : V \to \mathbb{Z}^+$, and let f be an r-degree specification. Suppose that G has a k-partition-connected r-detachment, then G has a k-partition-connected loopless r-detachment iff $r(v) \geq 2$ for all $v \in V$ with $i(v) \geq 1$.*
Suppose that G has a k-partition-connected f-detachment, then G has a k-partition-connected loopless f-detachment iff $d_i^v \leq d(v) + i(v)$ for all $v \in V$, $1 \leq i \leq r(v)$.

Theorem 5. *[2] Let $d = (d_1, d_2, ..., d_n)$ be a non-increasing sequence of non-negative integers with $n \geq 2$, $d_1 \leq d_2 + ... + d_n$, and for which $\sum_{i=1}^{n} d_i$ is even. Then there exists a loopless k-partition-connected graph with degree sequence d iff:*
i, $d_n \geq k$ and
ii, $\sum_{i=1}^{n} d_i \geq 2k(n-1)$

Proof. These conditions are necessary, because there exist k edge-disjoint spanning trees in the r-detachment. In the other direction, let G be a vertex v with $(\sum_{i=1}^{n} d_i)/2$ loops, $r(v) := n$ and $f(v) = (d_1, ...d_n)$. First observe that any $\mathcal{P}$ partition of V is $X_0 = v$, so $i(X_0) + e(\mathcal{P}) = (\sum_{i=1}^{n} d_i)/2$ and $k\cdot(t-1) + k\cdot r(X_0) = -k + kn = k(n-1)$, and because of the ii, condition, $i(X_0) + e(\mathcal{P}) \geq k\cdot(t-1) + k\cdot r(X_0)$ holds. From Theorem 3. we know there exists a k-partition-connected r-detachment, and also a k-partition-connected f-detachment, because for every $i = 1, ..., n$ $d_i \geq d_n \geq k$.
$d_i \leq d(v) + i(v) = 0 + (\sum_{i=1}^{n} d(i))/2$ holds, because $2d_i \leq 2d_1 \leq d_1 + d_2 + ... + d_n$. So the conditions of Theorem 4. holds and then there exists a k-partition-connected loopless f-detachment, which is the wanted graph. □

In [3] the main results are about the existence of 2-connected r-detachment (interesting, that in these theorems G is only 2-edge-connected). However, they also show conjectures about k-connected r-detachments. In [4] we want a (detached) graph G' which satisfies $\lambda_{G'}(x, y) \geq r(x, y)$, where $r : V \times V \to \mathbb{Z}^+$ is a requirement function.
There are detachments for directed graphs too. An r-degree specification of D is a function f on V, s.t. for each vertex $v \in V$, $f(v)$ is a sequence of ordered pairs $(\varrho_i^v, \delta_i^v)$, $1 \leq i \leq r(v)$ of positive integers so that $\sum_{i=1}^{r(v)} \varrho_i^v = \varrho(v) + i(v)$ and $\sum_{i=1}^{r(v)} \delta_i^v = \delta(v) + i(v)$. An f-detachment of D is an r-detachment in which the in- and out-degrees of the pieces of each $v \in V$ are given by the pairs of $f(v)$. There are also theorems about the existence of k-edge-connected r-detachment.

4 Conclusion

Since this technique is simple but useful, there are several applications of detachments in graph theory or in other part of science.

References

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