Graph Theory	**March 30, 2026**
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1 Introduction

Definition 1. In a graph G, a dominating set is a subset of vertices (usually denoted by D) such that every vertex in G is either an element of D or is adjacent to at least one member of D. The domination number of a graph $\gamma(G)$ is the size of the smallest dominating set.

Example 2. It is easy to see that $\gamma(K_n) = 1$, $\gamma(K_{m,n}) = 2$ and $\gamma(P_n) = \lceil \frac{n}{3} \rceil$.

2 Algorithms and computational complexity

Theorem 3. *For a given graph G and number k, it is NP-hard to determine whether $\gamma(G) \leq k$.*

Proof. We can do the following reduction: consider the Vertex Cover problem and an instance of it, denoted by (G, k). We create an instance (H, k) of the Dominating Set problem by deleting all isolated vertices and for all uv edges, adding a vertex uv connected only to u and v. A vertex cover of G is a dominating set of G and from the construction it follows easily that it is a dominating set of H too.
If a dominating set of H contains a vertex of type uv, we can replace it with either u or v. So every vertex of the dominating set is in G and since every vertex of type uv is dominated, for every edge in G at least one endpoint is in the dominating set, so it is indeed a vertex cover in G.

Theorem 4. *The greedy algorithm (starting with $S = \emptyset$ and adding vertices to S that increases the number of dominated vertices by the most) computes a $\ln \Delta + 2$-approximation.*

Proof. Once a vertex is added to S we can say that the cost (=size of the dominating set) is increased by 1. We can distribute this cost equally among all newly dominated vertices such that a $\frac{1}{\text{number of newly dominated nodes}}$ value is being assigned to each newly covered vertex.
Now let us take an S^* minimal dominating set. By assigning every vertex not contained by S^* to a neighboring node in S^* we can create stars, each of which having a cost of 1. In the following, we prove that the greedy (distributed) cost of each of these stars is at most $\ln \Delta + 2$. For a star with center v^* and let us take a vertex u which is the next one chosen to be in S by the greedy algorithm. Let $c(x)$ denote the number of vertices that become dominated once x is added to S at the point where the algorithm stands. If a vertex v from the star of v^* becomes dominated, its cost becomes $\frac{1}{c(u)}$. Since v^* is not in S yet, we know that $c(v^*) \leq c(u)$. Hence, the cost of v is at most $c(v^*)$. It follows that cost of the vertex covered first from the star of v^* is at most $\frac{1}{d(v^*)+1}$. Similarly, the cost of the vertex from the star that is covered as the second is at most $\frac{1}{d(v^*)}$, because when it is covered, $c(v^*) \geq d(v^*)$ holds. In general, the vertex that is covered for the i-th time, gets charged at most $\frac{1}{d(v^*)-i+2}$. It follows that the total cost of the star of v^* is at most

$$\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{d(v^*)+1} \leq \sum_{i}^{\Delta+1} \frac{1}{i} < ln(\Delta) + 2.$$

Theorem 5. *The minimum dominating set problem is* Log-APX-Complete.

Corollary 6. *The approximation of the greedy algorithm is optimal (up to lower-order terms).*

Theorem 7 (van Rooij, Nederlof, van Dijk)**.** *There exist an $O(1.5048^n)$ algorithm for the minimum dominating set problem.*

Remark 8. On trees, the domination number can be calculated in linear time using dynamic programming.

3 Classic properties and results

Claim 9. *If G has no isolated vertices, then $\gamma(G) \leq \frac{|V(G)|}{2}$. Also, for any G, $\gamma(G) \leq n - \Delta$.*

Proposition 10. *Every maximal independent set is dominating.*

Conjecture 11 (Vizing). *$\gamma(G \Box H) \geq \gamma(G)\gamma(H)$, where $G \Box H$ denotes the cartesian product of graphs G and H.*

4 Variants

Definition 12. An independent dominating set is a dominating set which is also (maximal) independent, and $i(G)$ denotes the minimum size of an independent dominating set.

Claim 13. *$\gamma(G) \leq i(G)$ and equality holds for claw-free graphs.*

Corollary 14. *The minimum maximal matching and the minimum edge dominating set of any graph have the same size.*

Definition 15. An independence dominating set of a graph G is a set that dominates every independent set of G, and $i\gamma(G)$ denotes the minimum size of an independence dominating set.

Claim 16. *For chordal graphs, $\gamma(G) = i\gamma(G)$ holds.*

Claim 17. *$\gamma(G \Box H) \geq i\gamma(G)\gamma(H)$ and $i\gamma(G \Box H) \geq i\gamma(G)i\gamma(H)$.*

Definition 18. A minimum connected dominating set of a graph G is a connected dominating set with the smallest possible cardinality among all connected dominating sets of G.

Claim 19. *If d is the connected domination number of an n-vertex graph G ($n > 2$), and l is the maximum number of leaves among all the spanning trees of G, then $d + l = n$ holds.*

References

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