1 Introduction

Definition 1. Let $G = (V, E)$ be an undirected graph. The line graph of G, denoted by $L(G)$, is a graph such that:

1. Each vertex of $L(G)$ represents an edge of G.
2. Two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G.

Properties of a graph G that depend only on adjacency between edges may be translated into equivalent properties in $L(G)$ that depend on adjacency between vertices (source: Wikipedia).

Example 2. The matchings of G correspond to the independent sets of $L(G)$ and vice versa. Therefore finding a maximum independent set in $L(G)$ can be done in polynomial time, despite the hardness of the problem.

Example 3. If G has an Euler cycle, then $L(G)$ is Hamiltonian. The implication does not work in the other direction.

2 Important Theorems

Theorem 4. *If G is connected, then $L(G)$ is also connected. If $L(G)$ is connected, then at most one of the connected components of G is not a vertex.*

Theorem 5. $|V(L(G))| = |E(G)|$ *and* $|E(L(G))| = \sum_{v \in V(G)} \binom{d_v}{2}$.

Theorem 6 (Rooij and Wilf [6])**.** *Consider the sequence $G, L(G), L(L(G)), \ldots$. If G is a finite connected graph, then only four behaviors are possible for this sequence:*

1. *If G is a cycle graph, then $L(G)$ and each subsequent graph in this sequence are isomorphic to G itself. (These are the only connected graphs for which $L(G)$ is isomorphic to G.)*
2. *If G is $K_{1,3}$, then $L(G)$ and all subsequent graphs in the sequence are triangles.*
3. *If G is a path graph then each subsequent graph in the sequence is a shorter path until eventually the sequence terminates with an empty graph.*
4. *In all remaining cases, the sizes of the graphs in this sequence eventually increase without bound.*

Theorem 7 (Whitney's Isomorphism Theorem [7])**.** *Let G and H be connected graphs. If $L(G) \cong L(H)$, then $G \cong H$, with exactly one exception: $G = K_{1,3}$ and $H = K_3$.*

Theorem 8. *Let G be a connected graph with at least 3 vertices. The line graph $L(G)$ is Eulerian if and only if all vertices of G have even degrees, or all vertices of G have odd degrees.*

3 Characterizations

Theorem 9 (Krausz's Characterization [3])**.** *A graph is a line graph if and only if its edges can be partitioned into cliques such that no vertex belongs to more than two of these cliques.*

Theorem 10 (Beineke's Theorem [1])**.** *A graph is a line graph if and only if it does not contain any of nine specific forbidden induced subgraphs.*

Remark 11. There exist linear time algorithms to decide if a given graph is a line graph or not.

4 Conjectures and Open Problems

Conjecture 1 (Matthews-Sumner [4])**.** *Every 4-connected claw-free graph is Hamiltonian.*

Conjecture 2 (Thomassen [5])**.** *Every 4-connected line graph is Hamiltonian.*

Conjecture 3 (Graham [2])**.** *Consider the sequence* $|V(G)|, |V(L(G))|, |V(L(L(G)))|, \ldots$. *If* G *is a tree, then it is uniquely determined up to isomorphism entirely by its corresponding sequence.*

References

[1] L. W. Beineke. Characterizations of derived graphs. *Journal of Combinatorial Theory*, 9(2):129–135, 1970.

[2] J. N. Cooper and B. Kay. Graham's tree reconstruction conjecture and a waring-type problem on partitions. *Journal of Combinatorics*, 9(3):471–483, 2018.

[3] J. Krausz. Démonstration nouvelle d'un théorème de whitney sur les réseaux. *Matematikai és Fizikai Lapok*, 50:75–85, 1943.

[4] M. M. Matthews and D. P. Sumner. Hamiltonian results in $k_{1,3}$-free graphs. *Journal of Graph Theory*, 8(1):139–146, 1984.

[5] C. Thomassen. Reflections on graph theory. *Journal of Graph Theory*, 10(3):309–324, 1986.

[6] A. C. M. van Rooij and H. S. Wilf. The interchange graph of a finite graph. *Acta Mathematica Academiae Scientiarum Hungaricae*, 16(3-4):263–269, 1965.

[7] H. Whitney. Congruent graphs and the connectivity of graphs. *American Journal of Mathematics*, 54(1):150–168, 1932.