Graph Theory	**Lecture date: March 16, 2026**
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1 Definition

In geometric graph theory, a **unit disk graph** (UDG) is the intersection graph of a family of unit disks in the Euclidean plane.

There are several equivalent definitions [1,3]:

- **Proximity model.** Given a set of points in $\mathbb{R}^2$, two vertices are connected by an edge if and only if their Euclidean distance is at most 2.
- **Intersection model.** Vertices correspond to equal-radius circles; two vertices are adjacent if and only if their circles have a non-empty intersection. (Tangent circles are considered to intersect.)

  WLOG, each radius is taken to be 1.
- **Containment variant.** Two equal-radius circles are connected whenever one circle's centre lies inside the other circle (the radii are taken to be 2 for the equivalence of the definitions).

There are many extensions of UDGs, for example (arbitrary radii) circle intersection graphs and unit regular polygon intersection graphs.

2 Applications

Unit disk graphs are a natural model whenever proximity implies interaction.

- **Wireless ad hoc networks.** Nodes with equal transmission power are modelled as points in the plane; two nodes are connected iff they are within range.
- **Frequency assignment.** Assigning distinct frequencies to transmitters with overlapping ranges corresponds to the *minimum coloring* problem on UDGs.
- **Facility location.** Placing k facilities where proximity is undesirable corresponds to finding a *maximum independent set* of size k.
- **Relay/base station selection.** Finding a minimum set of transmitters that cover all others corresponds to the *minimum dominating set* problem.

3 Structural Properties

Lemma 1. *Every induced subgraph of a UDG is also a UDG.*

Lemma 2 ($K_{1,6}$-free property)**.** *No unit disk graph contains $K_{1,6}$ as an induced subgraph.*

An important consequence: the maximum independent set in the neighbourhood of *any* vertex has size at most 5.

Lemma 3. *Let G be a UDG and let v be the vertex whose disk centre has the smallest x-coordinate. Then the maximum independent set in $G(N(v))$ has size at most* 3.

Lemma 4 (Clique lower bound)**.** *A UDG G with maximum degree Δ contains a clique of size at least* $\lceil \Delta/6 \rceil + 1$.

Further structural facts:

- UDGs are **not** perfect in general: an odd cycle C_k for $k \geq 5$ is a UDG but not perfect.
- UDGs are **not** planar in general: K_5 is a UDG.
- The number of UDGs on n labelled vertices is within an exponential factor of n^{2n}, implying UDGs do not have bounded twin-width.
- Infinitely many other forbidden induced subgraphs beyond $K_{1,6}$ are known.

4 Computational Complexity

Determining whether an abstract graph (given without geometry) is a UDG is **NP-hard**. Moreover, some UDGs require exponentially many bits of precision in any geometric realisation, making explicit coordinate output impossible in polynomial time.

The **maximum clique problem** can be solved exactly in polynomial time given a disk representation; and even without representation, a maximum clique or a proof of non-UDG membership can be produced in polynomial time.

Some classical problems remain **NP-hard** even on UDGs: Maximum independent set, Minimum vertex cover, Minimum vertex coloring, Minimum dominating set.

Here are some key heuristics for the approximation algorithms [2]:

- **Independent set (ratio 3).** Repeatedly pick the leftmost vertex v (smallest x-coord), add it to the independent set, and delete $v \cup N(v)$. Each deleted neighbourhood contains ≤ 3 independent vertices.
- **Dominating set (ratio 5).** Any maximal independent set is a dominating set; since UDGs are $K_{1,6}$-free, no dominator covers more than 5 others.
- **Online coloring (ratio 6).** Greedy first-fit uses $\leq \Delta + 1$ colors, and the $K_{1,6}$-free property implies a clique of size $\geq \lceil \Delta/6 \rceil + 1$, so OPT $\geq \Delta/6 + 1$.

The coloring problem admits a 3-approximation via on offline algorithm.

References

[1] B. N. Clark, C. J. Colbourn, and D. S. Johnson. Unit disk graphs. *Discrete Mathematics*, 86(1–3):165–177, 1990.

[2] M. V. Marathe, H. Breu, H. B. Hunt III, S. S. Ravi, and D. J. Rosenkrantz. Simple heuristics for unit disk graphs. *Networks*, 25(2):59–68, 1995.

[3] Wikipedia contributors. Unit disk graph. Wikipedia, The Free Encyclopedia. `https://en.wikipedia.org/wiki/Unit_disk_graph`.