Graph Theory	**Lecture date: March 30, 2026**
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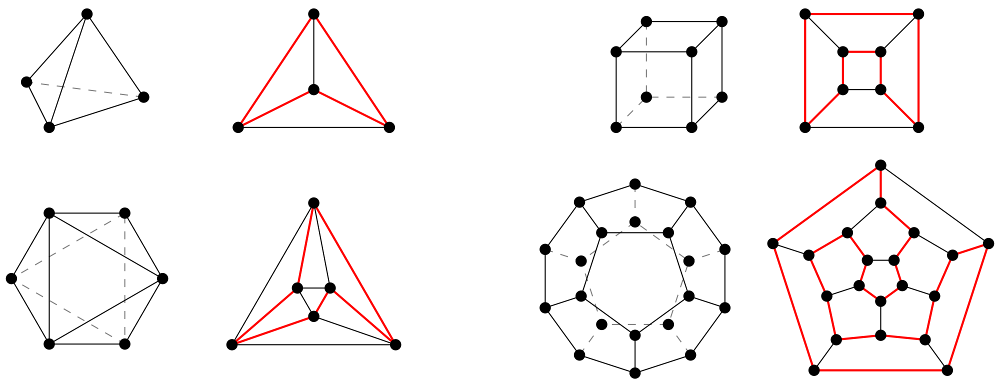
1 Introduction

Polyhedral graphs arise at the intersection of geometry and graph theory. Given a convex polyhedron, we can associate a graph whose vertices and edges correspond to those of the polyhedron. This leads to a fundamental question: *Which graphs arise as graphs of convex polyhedra?*

Definition 1. Let P be a convex polyhedron. The vertices and the edges of P form a finite, undirected, simple graph, called the graph of P and denoted by $G(P)$.



The analogue concept for d-polytopes of general dimension are the d-polytopal graphs.

2 Characterizations

Definition 2. The *face lattice* of a convex polytope P is the poset $L(P)$ of all faces of P, partially ordered by inclusion.

Definition 3. The relative interior $\operatorname{relint}(P)$ of a polytope is defined as the interior of P with respect to the embedding of P into its affine hull $\operatorname{aff}(P)$, in which P is full-dimensional.

Lemma 4. *Let $P \subset \mathbb{R}^d$ be a polytope. Then $P = \bigsqcup_{F \in L(P)} \operatorname{relint}(F)$, that is, P can be written as the disjoint union of the relative interiors of its faces.*

Lemma 5. *Let $v \in \operatorname{vert}(P)$ be a vertex, and let $N(v)$ be the set of its neighbors in $G(P)$. Then the cone (based at v) spanned by the neighbors of v contains P: $P \subseteq v + \operatorname{cone}\{u - v : u \in N(v)\}$.*

Theorem 6 (Balinski's theorem [3])**.** *The graph $G(P)$ is d-connected for every d-polytope P.*

Proof. Let $P = conv(V) \subseteq \mathbb{R}^d$, where the vertex set V of P (and of the graph $G(P)$) has at least $d+1$ elements. We have to show, that if we delete a subset of $d-1$ of them, $S = \{v_1, ..., v_{d-1}\} \subseteq V$; then the graph $G(P) \setminus S$ induced on the remaining vertices is connected.

Let $s = \frac{1}{d-1} \sum_{i=1}^{d-1} v_i \in P$ denote the barycenter of the vertex set S. We know by Lemma 4. that s is contained in the relative interior of a unique face F. We consider two cases:

Case 1. If s is contained in a proper face $F \in L(P) \setminus \{P\}$, then let $cx \leq c_0$ be a valid inequality that defines F. It is easy to see that all points $v_i \in S$ are also contained in this face F. The largest value that cx can achieve on P is c_0, while the smallest value is some $g_0 < c_0$. In this case, every vertex in $V \setminus S$ either lies in the face $F_0 = \{x \in P : cx = g_0\}$, or it has a neighbor whose cx-value is smaller (this follows from Lemma 5.), and which therefore also lies in $V \setminus S$. Thus every vertex in $V \setminus S$ has a decreasing path, within $V \setminus S$, which connects it to a vertex in F_0. Finally, the graph of F_0 is connected, by induction on d.

Case 2. If s is contained in the interior of P, then we choose a linear function cx such that the hyperplane $\{x \in \mathbb{R}^d : cx = c_0\}$ contains both S and at least one other vertex $v_0 \in V \setminus S$. This is possible because every set of d points is contained in a hyperplane.

Now let c_{max} and c_{min} denote the largest and the smallest value, respectively, that cx takes on P, and let F_{max} and F_{min} denote the corresponding faces. Then the graphs $G(F_{max})$ and $G(F_{min})$ are again connected, by induction. Every vertex $v \in V \setminus S$ is connected either by a strictly cx-increasing path which avoids S to F_{max} (if it satisfies $cv \geq c_0$), or by a strictly decreasing path to F_{min} (if $cv \leq c_0$). Finally, the extra vertex v_0 is connected to both F_{max} and F_{min}, so the whole graph $G(P) \setminus S$ is connected. □

Theorem 7 (Steinitz' theorem [3]). *G is the graph of a 3-dimensional polytope if and only if it is simple, planar, and 3-connected.*

This theorem has an interesting history, including the publication of a number of incomplete proofs. Its importance lies in the fact that it enables us to deduce properties of 3-polytopes from graph-theoretic results, and vice-versa. Several proofs of Theorem 7 are known, none of which is easy, and except for the partial result of Theorem 6, no extension to higher dimensions is known.

3 The Hirsch conjecture

Definition 8. The diameter of a graph G will be denoted by $\delta(G)$: the smallest number δ such that any two vertices in G can be connected by a path with at most δ edges.

Definition 9. For $n > d \geq 2$, let $\Delta(d, n)$ be the maximal diameter of the graph of an d-dimensional polytope P with at most n facets. For example, $\Delta(2, n) = \lfloor \frac{n}{2} \rfloor$.

It is a long-standing problem to determine the behavior of the function $\Delta(d, n)$. The value of $\Delta(d, n)$ is a lower bound for the number of iterations needed for the simplex algorithm with any pivot rule. Thus the question of whether $\Delta(d, n)$ grows polynomially in n and d is closely related to the question of whether there is any pivot rule for which the simplex algorithm is a strongly polynomial algorithm for linear programming.

Conjecture 1 (Hirsch conjecture [3]). *For $n > d \geq 2$, let $\Delta(d, n)$ denote the largest possible diameter of the graph of a d-polytope with n facets. Then $\Delta(d, n) \leq n - d$.*

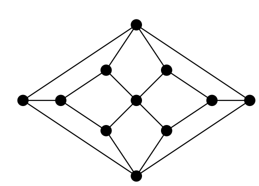
In 2010, Francisco Santos constructed a counterexample, which disproved the conjecture after more than 50 years of belief. Anyway, it holds for dimension $d \leq 3$, and also for greater dimensions the best known bounds are still not far from linear.

Theorem 10 (Kalai-Kleitman-Todd [2]). *For $1 \leq d \leq n$, $\Delta(d, n) \leq (n - d)^{log(d)}$, with $\Delta(1, 1) = 0$.*

4 Hamiltonian cycles

The general question whether each member of a certain class of graphs (for example, d-polytopal graphs) contains a subgraph of a specified type (for example, a Hamiltonian circuit, or a subdivision of a complete graph, etc.) arises in contexts as various as coding theory, linear programming, or the four-color problem. [1]

About Hamiltonian circuits Tait conjectured in 1880 that the graph of each simple 3-polytope admits a Hamiltonian circuit. If such a graph possesses a Hamiltonian circuit then the countries of the corresponding map (that is, the 2-faces of the 3-polytope) are colorable with 4 colors. In 1946 Tutte found a counterexample (with 46 vertices) to Tait's conjecture.



The smallest nonnamiltonian polyhedral graph is the Herschel graph.

Theorem 11. *Each* 4*-connected planar graph admits a Hamiltonian circuit.*

Theorem 12. *For every* $d \geq 3$ *there exist* d*-polytopal graphs which have no Hamiltonian circuit.*

Theorem 13. *Each* 3*-polytopal graph admits a spanning tree of maximal degree 3.*

Theorem 14. *The graph of each* d*-polytope contains as subgraph a subdivision of the complete graph with* $d+1$ *vertices.*

References

[1] B. Grünbaum. Polytopes, graphs, and complexes. *Bulletin of the American Mathematical Society*, 76:1131–1201, 1970.

[2] M. J. Todd. An improved kalai–kleitman bound for the diameter of a polyhedron. *SIAM Journal on Discrete Mathematics*, 28(4):1944–1947, 2014.

[3] G. M. Ziegler. *Lectures on Polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer, New York, 1995.