Matroid theory
Exercise Sheet 5
Date: 10 March 2026

Exercise 5.1. Which matroids can be obtained as the homomorphic image of a partition matroid?

Exercise 5.2. Show that in a bipartite graph $G = (S, T; E)$, if $X \subseteq S$ can be covered by a matching and $Y \subseteq T$ can be covered by a matching, then there exists a matching that covers $X \cup Y$.

Exercise 5.3. Let $G = (S, T; E)$ be a bipartite graph. Show that the direct sum of the transversal matroids defined on S and T is the matching matroid of G.

Exercise 5.4. **a)** Let $D = (V, A)$ be a directed graph with root r and let v be an arbitrary vertex. Show that the subsets of edges entering v, for which there exist edge-disjoint r-v paths using those edges, form the independent sets of a matroid.
b) In a directed graph $D = (V, A)$, a subset $F \subseteq A$ is called a *flame* if, for every vertex v, the number of edges in F entering v equals the number of pairwise edge-disjoint r-v paths in F. A *maximal flame* F is a flame with the additional property that for every vertex v, the number of edges in F entering v equals the maximum number of pairwise edge-disjoint r-v paths in D. Given a directed graph $D = (V, A)$ and a weight function $w \colon A \to \mathbb{R}$. Give a polynomial-time algorithm to find a minimum-cost maximum flame in an acyclic graph.

Homework (submission deadline: March 17)

Exercise 5.5. Prove that the class of paving matroids is closed under taking minors.

Challenging problem (submission deadline: March 31)

Exercise 5.6. Let $B = \{s_1, \ldots, s_r\}$ and $B' = \{s'_1, \ldots, s'_r\}$ be disjoint bases of the same transversal matroid. Prove that there is a permutation $(s_{\pi(1)}, \ldots, s_{\pi(r)})$ of the elements of B and a permutation $(s'_{\pi'(1)}, \ldots, s'_{\pi'(r)})$ of the elements of B' such that the sequence $(s_{\pi(1)}, \ldots, s_{\pi(r)}, s'_{\pi'(1)}, \ldots, s'_{\pi'(r)})$ is a cyclic ordering in which any r cyclically consecutive elements form a basis.

Research opportunity

Conjecture 5.7. Let $B = \{s_1, \ldots, s_r\}$ and $B' = \{s'_1, \ldots, s'_r\}$ be disjoint bases of the same matroid. Prove that there is a permutation $(s_{\pi(1)}, \ldots, s_{\pi(r)})$ of the elements of B and a permutation $(s'_{\pi'(1)}, \ldots, s'_{\pi'(r)})$ of the elements of B' such that the sequence $(s_{\pi(1)}, \ldots, s_{\pi(r)}, s'_{\pi'(1)}, \ldots, s'_{\pi'(r)})$ is a cyclic ordering in which any r cyclically consecutive elements form a basis.