Matroid theory
Exercise Sheet 4
Date: 3 March 2026

Exercise 4.1. Which matroids can be obtained as the homomorphic image of a uniform matroid?

Exercise 4.2. Let k be a positive integer, $\{S_1, \dots, S_t\}$ be a partition of S, and $0 \leq f_i \leq g_i \leq |S_i|$ be integers for all $1 \leq i \leq t$. Prove that

$$\mathcal{B} \coloneqq \{X \subseteq S \colon |X| = k,\ f_i \leq |X \cap S_i| \leq g_i \text{ for all } 1 \leq i \leq t\}$$

is either empty or forms the family of bases of a matroid.

Exercise 4.3. Prove that the dual of a rank r paving matroid M is paving if and only if M has a hypergraph representation in which each hyperedge has size r.

Exercise 4.4. An undirected graph $G = (V, E)$ contains k pairwise edge-disjoint spanning trees if and only if for any partition $\{V_1, \dots, V_t\}$ of the vertex set into nonempty sets, there are at least $k(t-1)$ edges with endpoints in different partition classes.

Exercise 4.5. Let S be a ground set of size at least r, $\mathcal{H} = \{H_1, \dots, H_q\}$ be a (possibly empty) collection of subsets of S, and $r, r_1, \dots, r_q$ be non-negative integers satisfying

$$|H_i \cap H_j| \leq r_i + r_j - r \text{ for } 1 \leq i < j \leq q.$$

Prove that $\mathcal{I} = \{X \subseteq S \mid |X| \leq r,\ |X \cap H_i| \leq r_i \text{ for } 1 \leq i \leq q\}$ forms the independent sets of a matroid with rank function $r(Z) = \min\big\{r, |Z|, \min_{1 \leq i \leq q} \{|Z - H_i| + r_i\}\big\}$.

Homework (submission deadline: March 10)

Exercise 4.6. Prove that a matroid M is connected if and only if M^* is connected.

Challenging problem (submission deadline: March 24)

Exercise 4.7. Let L/K be a field extension. We say that $\ell_1, \ell_2, \dots, \ell_k \in L$ are algebraically independent over K if there exists no nonzero polynomial $P \in K[x_1, x_2, \dots, x_k]$ such that $P(\ell_1, \ell_2, \dots, \ell_k) = 0$. Prove that the algebraically independent subsets of a finite set $S \subseteq L$ form the family of independent sets of a matroid. These matroids are called *algebraic matroids*.

Research opportunity

Question 4.8. Is it true that the dual of any algebraic matroid is also algebraic?