Matroid theory
Submodular or not?
Date: 24 February 2026

Decide whether each of the following set functions is always submodular or not. Unless stated otherwise, S denotes a finite, nonempty set, and $G = (V, E)$ is a finite, simple, connected graph.

(S1) $S = \{\text{chocolate (C), brownie (B), panna cotta (P)}\}$, $\varphi\colon 2^S \to \mathbb{R}$; $\varphi(\emptyset) = 0$, $\varphi(C) = 4$, $\varphi(B) = 8$, $\varphi(P) = 9$, $\varphi(C \cup B) = 11$, $\varphi(C \cup P) = 12$, $\varphi(B \cup P) = 15$, $\varphi(C \cup B \cup P) = 16$.

(S2) $\varphi\colon 2^S \to \mathbb{R}$; $\varphi(X) = 0$.

(S3) $S = \{1, 2, \ldots, n\}$, $\varphi\colon 2^S \to \mathbb{R}$; $\varphi(X) = \max\{x : x \in X\}$, if $X \neq \emptyset$, $\varphi(\emptyset) = 0$.

(S4) $S = \{1, 2, \ldots, n\}$, $\varphi\colon 2^S \to \mathbb{R}$; $\varphi(X) = \min\{x : x \in X\}$, if $X \neq \emptyset$, $\varphi(\emptyset) = 0$.

(S5) $S = \{1, 2, \ldots, n\}$, $\varphi\colon 2^S \to \mathbb{R}$; $\varphi(X) = \sum_{i \in X} i$.

(S6) $\varphi\colon 2^S \to \mathbb{R}$; $\varphi(X) = \sqrt{|X|}$.

(S7) $\varphi\colon 2^S \to \mathbb{R}$; $\varphi(X) = \min\{3, \sqrt{|X|}\}$.

(S8) $M = (S, r)$ is a matroid, $\varphi\colon 2^S \to \mathbb{R}$; $\varphi(X) = r(X)$.

(S9) $\varphi\colon 2^V \to \mathbb{R}$; $\varphi(X) = \big|\{e \in E : \text{ exactly one of the endpoints of } e \text{ is in } X\}\big|$.

(S10) $\varphi\colon 2^V \to \mathbb{R}$; $\varphi(X) = \big|\{e \in E : \text{ both of the endpoints of } e \text{ are in } X\}\big|$.

(S11) $\varphi\colon 2^V \to \mathbb{R}$; $\varphi(X) = \big|\{e \in E : \text{ at least one of the endpoints of } e \text{ is in } X\}\big|$.

(S12) $\varphi\colon 2^V \to \mathbb{R}$; $\varphi(X) = \big|\{e \in E : \text{ at most one of the endpoints of } e \text{ is in } X\}\big|$.

(S13) $\varphi\colon 2^V \to \mathbb{R}$; $\varphi(X) = \big|\{v \in V : v \in X \text{ or } v \text{ has a neighbor in } X\}\big|$.

(S14) $\varphi\colon 2^E \to \mathbb{R}$; $\varphi(X) = c(X)$ where $c(X)$ is the number of connected components of the graph $G_X = (V, X)$.

(S15) $S = \{1, 2, \ldots, n\}$, $A_1, A_2, \ldots, A_n$ are finite sets, $\varphi\colon 2^S \to \mathbb{R}$; $\varphi(X) = \big|\bigcup_{i \in X} A_i\big|$.

(S16) $t \in V$, $S = V - t$, $c\colon E \to \mathbb{R}_+$ is a capacity function, $\varphi\colon 2^S \to \mathbb{R}$; $\varphi(X) =$ max value of an $X - t$ flow.

(S17) $\varphi\colon 2^E \to \mathbb{R}$; $\varphi(X) = (-1)$ times the number of triangles in X.

(S18) $S = \{1, 2, \ldots, n\}$, $A_1, A_2, \ldots, A_n$ are discrete random variables, $\varphi\colon 2^S \to \mathbb{R}$; $\varphi(X) = H(A_X)$, where $H(A_X)$ is the joint entropy of $\{A_i\}_{i \in X}$.

(S19) Let $b\colon 2^S \to \mathbb{Z}_+$ be an increasing submodular set function with $b(\emptyset) = 0$, $\varphi\colon 2^S \to \mathbb{R}$; $\varphi(X) = \min\{b(Y) + |X - Y| \colon Y \subseteq X\}$.

(S20) $\varphi\colon 2^V \to \mathbb{R}$; $\varphi(X) =$ the expected number of steps before a random walk hits X (started at a uniform random vertex, and moving uniformly at random to a neighbor in each step).