Matroid theory
Exercise Sheet 3
Date: 24 February 2026

Exercise 3.1. **a)** Prove that $(S9)$ is always submodular, or provide a counterexample.
b) Prove that $(S19)$ is always submodular, or provide a counterexample.

Exercise 3.2. Let $M = (S, \mathcal{F})$ be a matroid and $Z_1 \subseteq Z \subseteq S$, $Z_2 \coloneqq Z - Z_1$. Prove that $M/Z = (M/Z_1)/Z_2$.

Exercise 3.3. Let M be a matroid. Prove that the truncation and elongation operations are dual to each other in the sense that the dual of a truncation of M is an elongation of M^*.

Exercise 3.4. If B_1 and B_2 are bases of a matroid and $x \in B_1 - B_2$ then there exists $y \in B_2 - B_1$ such that $B_1 - x + y$ and $B_2 - y + x$ are both bases.

Exercise 3.5. **a)** Let S be a finite set and $\varphi \colon 2^S \to \mathbb{R}$ be a set function. Prove that φ is submodular if and only if, for every $s \in S$, the set function $\varphi_s \colon 2^{S-s} \to \mathbb{R}$ defined by $\varphi_s(X) = \varphi(X+s) - \varphi(X)$ is decreasing, i.e., $\varphi_s(X) \geq \varphi_s(Y)$ whenever $X \subseteq Y$.
b) Prove that φ is submodular if and only if, for every $s_1, s_2 \in S$ and every $X \subseteq S$, it holds that $\varphi(X) + \varphi(X + s_1 + s_2) \leq \varphi(X + s_1) + \varphi(X + s_2)$.

Exercise 3.6. **a)** Let $S = \{1, 2, \ldots, n\}$, and let $\varphi : 2^S \to \mathbb{Z}_+$ be an increasing submodular set function with $\varphi(\emptyset) = 0$ (i.e., an integer polymatroid). Prove that there exist a finite set T, a matroid $M = (T, r)$, and a partition $T = T_1 \cup T_2 \cup \ldots \cup T_n$ such that $\varphi(X) = r\big(\bigcup_{i \in X} T_i\big)$ for all $X \subseteq S$.
b) Let $S = \{1, 2, \ldots, n\}$, and let $\varphi : 2^S \to \mathbb{R}_+$ be an increasing submodular set function with $\varphi(\emptyset) = 0$ (i.e., a polymatroid). Prove that for every $\varepsilon > 0$ there exist a finite set T, a constant $c \in \mathbb{R}_+$, a matroid $M = (T, r)$, and a partition $T = T_1 \cup T_2 \cup \ldots \cup T_n$ such that $\left|\varphi(X) - c \cdot r\left(\bigcup_{i \in X} T_i\right)\right| \leq \varepsilon$ for all $X \subseteq S$.

Homework (submission deadline: March 3)

Exercise 3.7. Let $M = (S, \mathcal{F})$ be a matroid, let $w \colon S \to \mathbb{R}$ be a weight function and let $F \subseteq S$. Give a polynomial-time algorithm (using an independence oracle) to determine whether there exists a maximum-weight basis that contains F.

Challenging problem (submission deadline: March 17)

Exercise 3.8. Let $D = (V, A)$ be a finite directed graph (which can contain loops). Call a subset $S \subseteq V$ *unmeetable* if the following property holds. There exists an infinitely long directed walk $s = v_0^s, v_1^s, v_2^s, \ldots$ for all $s \in S$ such that $v_n^s \neq v_n^t$ for all distinct $s, t \in S$ and for all $n \in \mathbb{Z}_+$. Prove that the unmeetable sets are the independent sets of a matroid for every digraph D, called the *no-meet matroid* of D.

Research opportunity

Question 3.9. Given a matroid with an independence oracle. Is it possible to determine whether it is a no-meet matroid?