Matroid theory
Exercise Sheet 2
Date: 17 February 2026

Exercise 2.1. Assume that $r : 2^S \to \mathbb{Z}_+$ satisfies the rank axioms, i.e.,
(R1) $r(\emptyset) = 0$,
(R2) $r(X) \leq r(Y)$ if $X \subseteq Y$,
(R3) $r(X) \leq |X|$,
(R4) $r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y)$.
Let

$$\mathcal{F} := \{F \subseteq S : r(X) = |X|\}.$$

Prove that $\mathcal{F}$ is the family of independent sets of a matroid with rank function r.

Exercise 2.2. Characterize all graphs G for which the associated graphic matroid $M(G)$ is connected.

Exercise 2.3. **a)** Prove that the circuit oracle can be generated by the independence oracle in polynomial steps.
a) Prove that the independence oracle cannot be generated by the circuit oracle in polynomial steps.

Exercise 2.4. Prove that if two matroids are given on the ground set S whose rank functions satisfy $r_1(X) + r_2(S \setminus X) \geq k$ for every $X \subseteq S$, then $\hat{r}_1(c) + \hat{r}_2(\chi_S - c) \geq k$ also holds for every $c : S \to [0, 1]$.

Exercise 2.5. Consider the following axiom (G):
For every weight function $c : S \to \mathbb{R}$, the greedy algorithm produces a maximal member of $\mathcal{F}$ of maximal weight. Prove that the sets of axioms $\{(I1), (I2), (I3)\}$ and $\{(I1), (I2), (G)\}$ are equivalent.

Exercise 2.6. Let $M = (S, \mathcal{F})$ be a matroid and $c : 2^S \to \mathbb{R}$ be a weight function. Prove that a basis B has maximal weight if and only if for all $y \in S - B$ and $x \in C(B, y)$ we have $c(y) \leq c(x)$.

Exercise 2.7. Let C_1 and C_2 be two circuits, $e \in C_1 \cap C_2$ and $f \in C_1 - C_2$. Prove that there exists a circuit C with $f \in C \subseteq C_1 \cup C_2 - e$.

Homework (submission deadline: February 24)

Exercise 2.8. Prove that if C is a circuit and $e \in C$ then there exists a basis B such that $C = C(B, e)$.

Challenging problem (submission deadline: March 10)

Exercise 2.9. **a)** Consider the Bingo game from class. Let $\mathcal{C}$ be a family of subsets of a finite set S with the following property: for every ordering $s_1, s_2, \ldots, s_n$ of the elements of S, if i is the smallest index such that the prefix set $\{s_1, \ldots, s_i\}$ contains a member of $\mathcal{C}$, then $\{s_1, \ldots, s_i\}$ contains exactly one member of $\mathcal{C}$. Moreover, for every $C \in \mathcal{C}$ there exists an ordering for which C is the unique member of $\mathcal{C}$ contained in the corresponding prefix $\{s_1, \ldots, s_i\}$. Prove that the families $\mathcal{C}$ with these properties are precisely the circuit families of matroids on S.
b) Prove that the family of subsets $I \subseteq S$ that contain no member of $\mathcal{C}$ is a family of independent sets of a matroid.
Use only results proved in lecture or in the exercise class.

Research opportunity

Question 1. *Can this alternative* Bingo *characterization of matroids be used to re-derive familiar matroid concepts and proofs in a more transparent way, and/or suggest meaningful generalizations beyond the matroid setting?*