Matroid theory
Exercise Sheet 1
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Exercise 1.1. Consider the following axiom $(I3'')$:
If $I_k, I_{k+1} \in \mathcal{F}$ with $|I_k| = k$ and $|I_{k+1}| = k+1$ then there exists $s \in I_{k+1} - I_k$ such that $I_k + s \in \mathcal{F}$.
Prove that the sets of axioms $\{(I1), (I2), (I3)\}$ and $\{(I1), (I2), (I3'')\}$ are equivalent.

Exercise 1.2. Let $C_1, \ldots, C_k$ be pairwise disjoint circuits in a matroid M, and let $x_i \in C_i$ for $i = 1, \ldots, k$. Furthermore, let C be a circuit of M distinct from all C_is. Verify that M has a circuit that is disjoint from $\{x_1, \ldots, x_k\}$.

Exercise 1.3. Prove that if $X \subseteq Y \subseteq S$, and there exist bases $B_1 \supseteq X$ and $B_2 \subseteq Y$ then there exists a bases $X \subseteq B_3 \subseteq Y$.

Exercise 1.4. Let $\mathrm{cl}(A)$ be the closure of A, i.e. $\mathrm{cl}(A) = \{s \in S \ : \ r(A+s) = r(A)\}$. Prove that $r(\mathrm{cl}(A)) = r(A)$.

Exercise 1.5. Assume that $\mathcal{B}$ satisfies the basis axioms, i.e.,

(B1) $\mathcal{B}$ is nonempty,

(B2) if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$ then there exists $y \in B_2 - B_1$ such that $B_1 - x + y \in \mathcal{B}$.

Let

$$\mathcal{F} := \{F \subseteq S \colon \text{there exists } B \in \mathcal{B} \text{ with } F \subseteq B\}.$$

Prove that $\mathcal{F}$ is the family of independent sets of a matroid, and the family of bases of this matroid is $\mathcal{B}$.

Exercise 1.6. Prove that for any bases B_1 and B_2 of a matroid there exists a bijection $f : B_1 - B_2 \to B_2 - B_1$ such that $B_1 - x + f(x)$ is a bases for all $x \in B_1 - B_2$.

Exercise 1.7. Let $G = (V, E)$ be a connected undirected graph.

a) Give a polynomial-time algorithm to determine a minimum sized subset $F \subseteq E$ intersecting every spanning tree of G.

b) Let $w \colon E \to \mathbb{R}$ be a weight function. Give a polynomial-time algorithm to determine a minimum sized subset $F \subseteq E$ intersecting every maximum weight spanning tree of G.

Homework (submission deadline: February 17)

Exercise 1.8. For $(M6)$, $(M9)$, $(M13)$, $(M20)$, and $(M30)$, either prove that it always yields a matroid or provide a counterexample.
Each problem is worth two points; at most three may be submitted.

Challenging problem (submission deadline: March 3)

Exercise 1.9. Let $G = (V, E)$ be a graph with $|V| = n$ such that E can be decomposed into two disjoint spanning trees. Prove that there exists a bijection $\varphi \colon E \to \{1, \ldots, 2n-2\}$ for which every cycle of G contains two edges with consecutive numbers.

Research opportunity

Conjecture 1.10. Let $G = (V, E)$ be a graph with $|V| = n$ such that E can be decomposed into two disjoint spanning trees. Prove that there exists a bijection $\sigma \colon E \to \{1, \ldots, 2n-2\}$ for which every cut of G contains two edges with consecutive numbers.