Deep learning and continuous optimization

Spring semester 2025/26

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General information

- **Objective:** To give an overview of basic techniques and results. An ideal outcome is that you can use these ideas in your work after this course.
  - basics of linear programming, LP solving techniques, integer programming, convex sets and functions, convex optimization
- **Course structure: 6** lectures and problem solving sessions
- **Course requirement:** 50% midterm (max 50 pts), 50% homework (max 50pts)
- **Evaluation:** 0-39 – 1, 40-54 – 2, 55-69 – 3, 70-84 – 4, 85-100 – 5
- **Contact:** `kristof.berczi@ttk.elte.hu`, Room 3.502
- **Reading:**
  - D. Bertsimas, J.N. Tsitsiklis. Introduction to linear optimization.
  - N. Vishnoi. Algorithms for convex optimization.
  - L.C. Lau. Convexity and optimization.
  - S. Bubeck. Convex Optimization: Algorithms and Complexity.
  - S. Boyd, L. Vandenberghe. Convex Optimization.

Lecture 1: Linear programming

Systems of linear equations

Example: A firm produces two different goods using two different raw materials. The available amounts of materials are 12 and 5, respectively. The goods require 2 and 3 units of the first material, and both require 1 unit of the second material. Find a production plan that uses all the raw materials.

Idea: Let x_1 and x_2 denote the amounts of the first and second goods produced, respectively. Then the constraints can be written as

$$2 \cdot x_1 + 3 \cdot x_2 = 12$$
$$x_1 + x_2 = 5$$

$$\begin{array}{cc} x_1 & x_2 \end{array}$$
$$\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 5 \end{bmatrix}$$

Solution

Step 1. $x_1 = 5 - x_2$
Step 2. $10 - 2 \cdot x_2 + 3 \cdot x_2 = 12 \Rightarrow x_2 = 2$
Step 3. $x_1 = 5 - 2 = 3$

In general

In general: Gauss elimination

$$\begin{array}{|cccc|}\hline x_1 & x_2 & \dots & x_n \\ \hline\end{array}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & a'_{12} & \dots & a'_{1n} \\ 0 & 1 & \dots & a'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a'_{mn} \end{bmatrix} = \begin{bmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_m \end{bmatrix}$$

Reduction of the matrix using **elementary row operations**, such as

- swapping two rows,
- multiplying a row by a nonzero number,
- adding a multiple of a row to another row.

Remarks:

- The set of solutions does not change.
- A final solution is ‘easy’ to read out.

Existence of a solution

Assume that your boss gives you such a problem, that is, solve $Ax = b$.

How to prove that a solution exists?

- Just provide a solution x.

How to prove that there is **no** solution?

- Gauss elimination concludes whether there exists a solution or not.
  BUT: this requires the understanding of the algorithm (that you cannot necessarily assume about your boss...)
- Would it be possible to provide some 'shorter' proof?

Fredholm alternative theorem

Fredholm alternative theorem

There exists an x satisfying $Ax = b$ if and only if there exists no y such that $yA = 0$, $yb \neq 0$.

Proof of 'only if' direction.

We show that at most one of x and y may exist. Suppose to the contrary that x and y are such that $Ax = b$ and $yA = 0, yb \neq 0$. Then

$$0 = (yA)x = y(Ax) = yb \neq 0,$$

a contradiction. □

Conclusion: The non-existence of a solution can be proved by providing y.

Geometric interpretation

Naming convention:

Primal problem

$$Ax = b \quad \textbf{(P)}$$

Dual problem

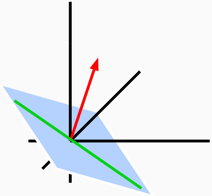
$$\begin{aligned} yA &= 0 \\ yb &\neq 0 \end{aligned} \quad \textbf{(D)}$$

$\Rightarrow$ Fredholm's theorem states that exactly one of **(P)** and **(D)** has a solution.

- The set $H := \{x : ax = b\}$ is a **hyperplane**.
- y is a **normal vector** of the hyperplane $H = \{x : ax = b\}$ if $yx = 0$ for every $x \in H$.

Fredholm as separation theorem

Either b lies in the subspace generated by the columns of A, or it can be separated from it by a homogeneous hyperplane with normal vector y.



The diet problem

What happens if, instead of equalities, a system of linear inequalities is given?

Example: A list of available foods is given together with the nutrient content. Furthermore, the requirement per day of each nutrient is also prescribed. For example, the data corresponding to two types of fruits (F1 and F2) and three types of nutrients (fats, proteins, vitamins) is as follows:

	Fats	Proteins	Vitamins	Available
F1	1	4	5	3
F2	0	2	9	5
Req.	1	5	14	

The problem is to find how much of each fruit to consume per day so as to get the required amount per day of each nutrient, if one can consume at most 2 kg of fruits per day.

Modeling the problem

	Fats	Proteins	Vitamins	Available
F1	1	4	5	3
F2	0	2	9	5
Req.	1	5	14	

Let x_1 and x_2 denote the amounts of fruits F1 and F2 to be consumed per day.

$$\begin{aligned} x_1 \quad\quad &\geqslant 1 \\ 4x_1 + 2x_2 &\geqslant 5 \\ 5x_1 + 9x_2 &\geqslant 14 \\ x_1 + x_2 &\leqslant 2 \end{aligned}$$

Questions:

- How to decide feasibility?
- How to find a solution (if exists) algorithmically?
- How to verify that there is no solution?

Different forms

Observations:

- An equality $ax = b$ can be represented as a pair of inequalities $ax \leqslant b$ and $-ax \leqslant -b$.
- An inequality $ax \leqslant b$ can be represented as the combination of an equality $ax + s = b$ and a non-negativity constraint $s \geqslant 0$, where s is called a **slack** variable.
- A non-positivity constraint $x \leqslant 0$ can be expressed as a non-negativity constraint $-x \geqslant 0$.
- A variable x unrestricted in sign can be replaced everywhere by $x^+ - x^-$, where $x^+, x^- \geqslant 0$.

General form

$$Px_0 + Ax_1 = b_0$$
$$Qx_0 + Bx_1 \leqslant b_1$$
$$x_1 \geqslant 0$$

Standard form

$$Ax = b$$
$$x \geqslant 0$$

Canonical form

$$Qx \leqslant b$$
$$x \geqslant 0$$

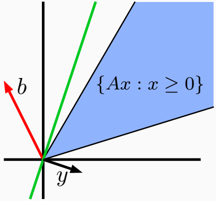
Farkas' lemma

Farkas' lemma, standard form

There exists an x satisfying **(P)** $Ax = b, x \geqslant 0$ if and only if there exists no y such that **(D)** $yA \geqslant 0, yb < 0$.

Farkas' lemma as separation theorem

Either b lies in the cone generated by the columns of A, or it can be separated from it by a homogeneous hyperplane with normal vector y.



Proof of the 'only if' direction.

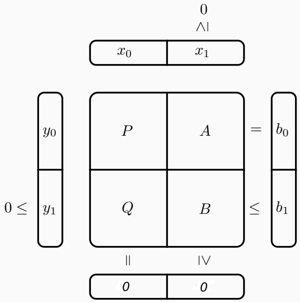
We show that at most one of x and y may exist. Suppose to the contrary that x and y are such that $Ax = b, x \geqslant 0$ and $yA \geqslant 0, yb < 0$. Then

$$0 \leqslant (yA)x = y(Ax) = yb < 0,$$

a contradiction. □

Farkas' lemma, general form

There exists an $x = (x_0, x_1)$ satisfying
(P) $Px_0 + Ax_1 = b_0, Qx_0 + Bx_1 \leqslant b_1, x_1 \geqslant 0$
if and only if there exists no $y = (y_0, y_1)$ such that
(D) $y_0 P + y_1 Q = 0, y_0 A + y_1 B \geqslant 0, y_1 \geqslant 0, y_0 b_0 + y_1 b_1 < 0$.



Conclusion: The feasibility/infeasibility of a system of linear inequalities can be proved by providing a solution to the primal/dual problem, respectively.

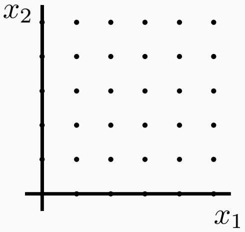
Remaining question: How to find such a solution?

$\Rightarrow$ We will answer this in a far more general setting!

Geometric background I

Example

$$\begin{aligned} x_1 + 2 \cdot x_2 &\leqslant 8 \\ 2 \cdot x_1 + \quad x_2 &\leqslant 6 \\ x_1, x_2 &\geqslant 0 \end{aligned}$$



Geometric background I

Example

$$x_1 + 2 \cdot x_2 \leqslant 8$$
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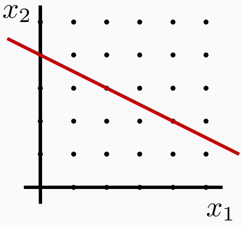

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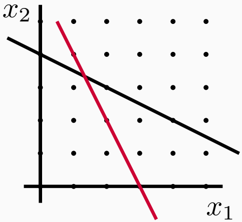

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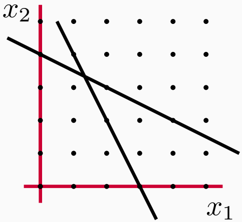

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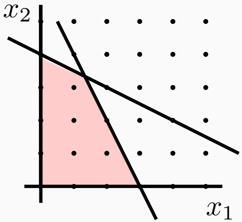

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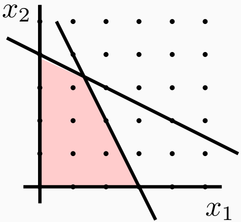
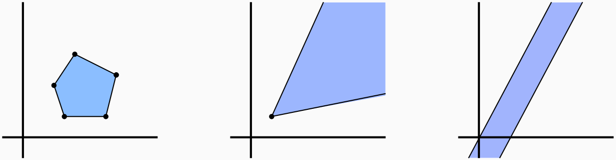


- An inequality $ax \leqslant b$ defines a **half space**.
- The solution set is the **intersection** of a finite number of half spaces, called a **polyhedron**.

Geometric background II



- Given a polyhedron P, a point $x \in P$ is a **vertex** of P if there exists no y such that $x + y, x - y \in P$.
- A **polytope** is the convex hull of a finite number of points.

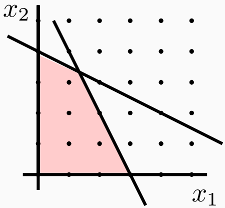
Thm.

Every polytope is a polyhedron, and every bounded polyhedron is the convex hull of its vertices.

Geometric background III

Example

$$\begin{aligned} x_1 + 2 \cdot x_2 &\leqslant 8 \\ 2 \cdot x_1 + \quad x_2 &\leqslant 6 \\ x_1, x_2 &\geqslant 0 \end{aligned}$$



Goal: Maximize/minimize a linear objective function over the set of solutions.

$\Rightarrow$ Example: $\max\{x_1 + x_2\}$.

Example

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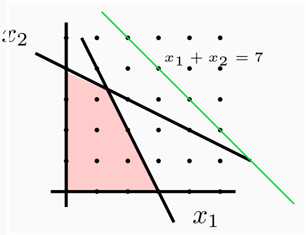

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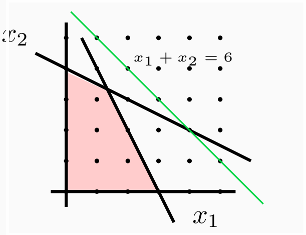

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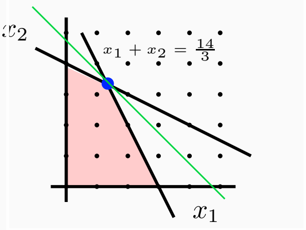

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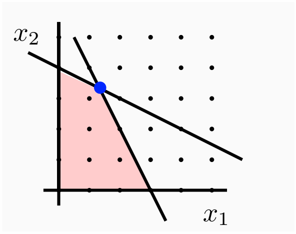

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Idea: Start from a vertex, and move to a neighboring vertex with improved objective value.

Geometric background III

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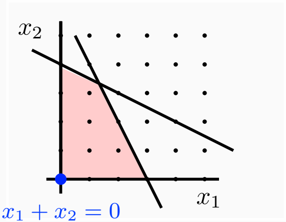

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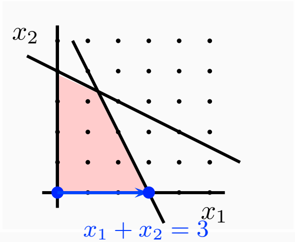

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Geometric background III

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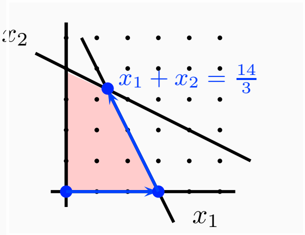

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History

1827, Fourier: Fourier-Motzkin elimination
1939, Kantorovich: reducing costs of army, general LP
1940's, Koopmans: economic problems as LPs
1941, Hitchcock: transportation problems as LPs
1946-47, Dantzig: general LP for planning problems in US Air Force (simplex method)
1979, Khachiyan: ellipsoid method, LP is solvable in linear time (more theoretical than practical)
1984, Karmakar: interior-point method (can be used in practice)

Linear programs

We would like to solve problems of the form

General form

$$\begin{aligned} \max\ & c_0x_0 + c_1x_1 \\ \text{s.t.}\ & Px_0 + Ax_1 = b_0 \\ & Qx_0 + Bx_1 \leqslant b_1 \\ & x_1 \geqslant 0 \end{aligned}$$

Standard form

$$\begin{aligned} \max\ & cx \\ \text{s.t.}\ & Ax = b \\ & x \geqslant 0 \end{aligned}$$

Canonical form

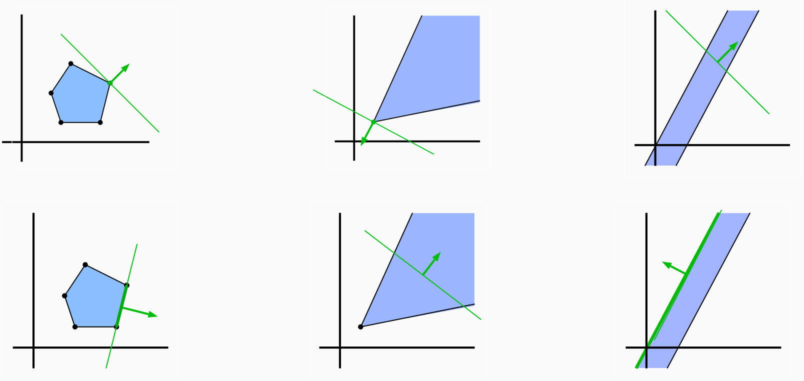
$$\begin{aligned} \max\ & cx \\ \text{s.t.}\ & Qx \leqslant b \\ & x \geqslant 0 \end{aligned}$$

Remarks:

- A minimization problem $\min\ cx$ can be reformulated as a maximization problem $\max\ (-c)x$ and vice versa.
- The optimal solution can be obtained by 'moving' a hyperplane with normal vector c towards the polyhedron, and finding the first point where they meet [**Be careful:** min or max?]

  $\Rightarrow$ Intuition: the optimum is always attained at a vertex.

Geometric background IV



Possible cases:

- single optimal solution,
- infinite number of optimal solutions, or
- no optimal solution (unbounded objective value).

Geometric background V

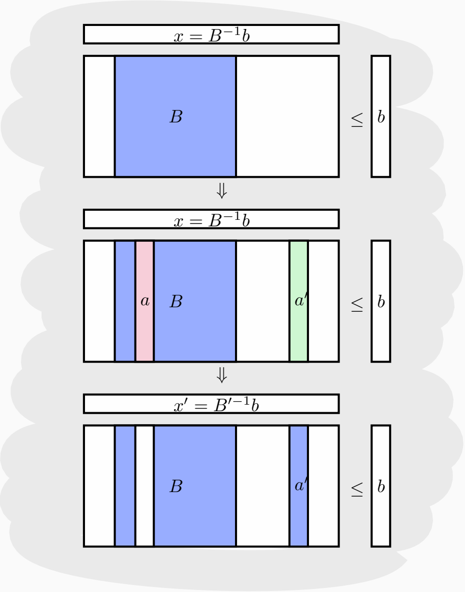
Thm.

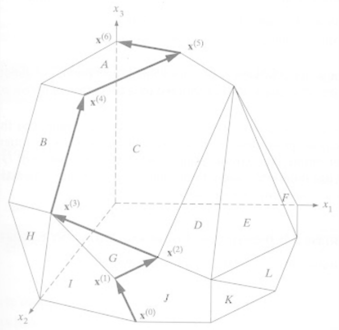
Let $P = \{x : Qx \leqslant b\}$ where the columns of Q are linearly independent. Then $x \in P$ is a vertex if and only if it can be obtained by taking a non-singular $r(Q) \times r(Q)$ submatrix Q' of Q and the corresponding part b' of b, and solving the system $Q'x = b'$.

Remarks:

- The number of such submatrices, and so the number of vertices is finite.
  $\Rightarrow$ If each vertex is visited at most once, then the procedure terminates.
- When the columns are non-independent, then there is an infinite number of basic feasible solutions. However, there are only a finite number of so-called **strong basic feasible solutions**, and, if it exists, the optimum is attained in one of them.

Simplex method





Problems

- Running time?
- Optimal solution?

Running time

Problem 1: The simplex algorithm might fail to terminate.

Reason: The algorithm can fall into cycles between bases associated with the same basic feasible solution and objective value.

Solution: Careful pivoting rule, e.g. **Bland's rule** prevents cycling.

Problem 2: Efficient in practice, but for almost every variant, there is a family of linear programs for which it performs **badly**.

Reason: The number of vertices of a polyhedron can be exponentially large.

Solution: Sub-exponential pivot rules are known.

Major open problem: Is there a variant with polynomial running time?

- **Hirsch's conjecture:** Let P be a d-dimensional convex polytope with n facets. Then the diameter of P is at most $n - d$.
- Counterexample by Francisco Santos, 2011 (86 facets, 43-dimensional).

Duality theorem

Problem 3: Is the solution optimal?

Duality theorem

Consider the problems

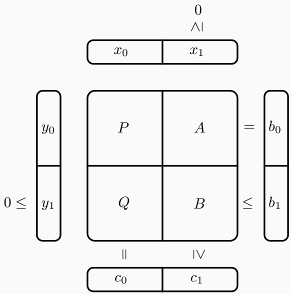
(P) $\max(c_0x_0 + c_1x_1)$ s.t. $Px_0 + Ax_1 = b_0, Qx_0 + Bx_1 \leqslant b_1, x_1 \geqslant 0$

and

(D) $\min(y_0b_0 + y_1b_1)$ s.t. $y_0P + y_1Q = c_0, y_0A + y_1B \geqslant c_1, y_1 \geqslant 0$.

Then exactly one of the followings hold:

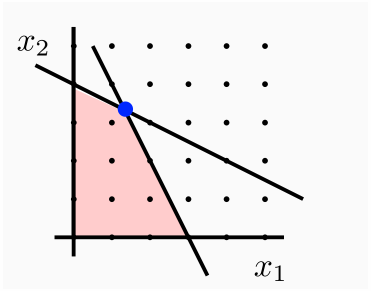
1. both (P) and (D) are empty,
2. (D) is empty and (P) is unbounded,
3. (P) is empty and (D) is unbounded,
4. both (P) and (D) have a solution, and $\max = \min$.



Example revisited

Example:

$$x_1 + 2 \cdot x_2 \leqslant 8$$
$$2 \cdot x_1 + x_2 \leqslant 6$$
$$x_1, x_2 \geqslant 0$$



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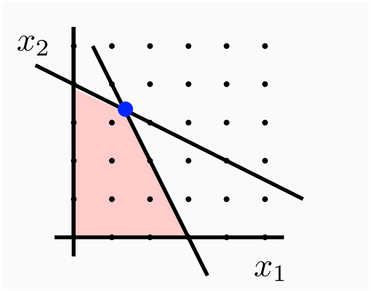

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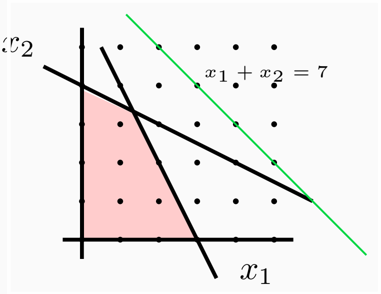

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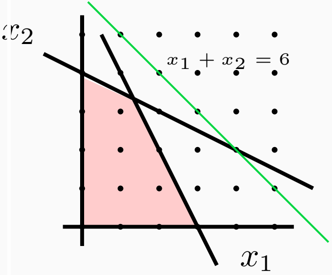
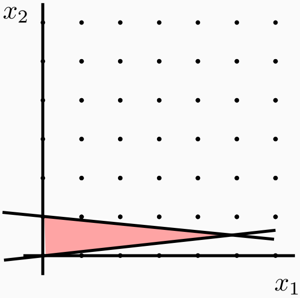


Another example

Example:

$$x_1 + 10 \cdot x_2 \leqslant 10$$
$$x_1 - 10 \cdot x_2 \leqslant 0$$
$$x_1, x_2 \geqslant 0$$
$$\max\{x_1\}$$



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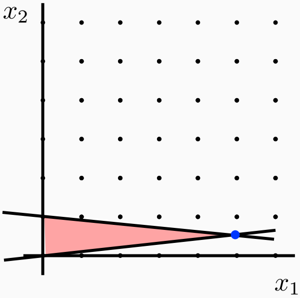

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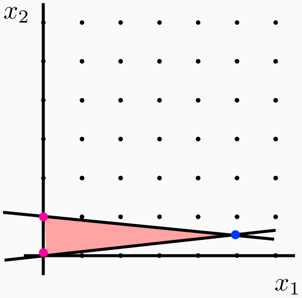
$$x_1, x_2 \geqslant 0$$

$$\max\{x_1\}$$



Another example

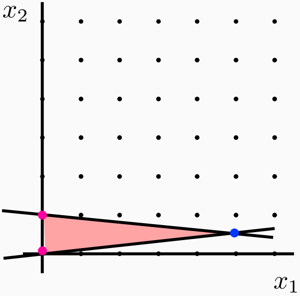
Example:

$$x_1 + 10 \cdot x_2 \leqslant 10$$

$$x_1 - 10 \cdot x_2 \leqslant 0$$

$$x_1, x_2 \geqslant 0$$

$$\max\{x_1\}$$



The fractional optimum can be far from the integer one.

Approaches

Bad news: integer programming is NP-complete

Approaches

Bad news: integer programming is NP-complete

Good news: there exist efficient algorithms

- totally unimodular matrices
  - every square submatrix has determinant 0, +1 or -1
- cutting plane methods
  - adding further inequalities that separate the actual optimum from the convex hull of the true feasible set
- branch and bound methods
  - systematically enumerating the candidate solutions, forming a rooted tree
- rounding methods (threshold rounding, iterative rounding)
  - rounding the coordinates of an optimal fractional solution
- heuristic methods (tabu search, hill climbing, simulated annealing, ant colony optimization, etc.)
  - some would call these 'voodoo'...

Branch and bound I

$$\min \; c(x) \quad \text{s.t. } x \in F$$

Here F is the set of integer feasible solutions to the problem.

Ideas:

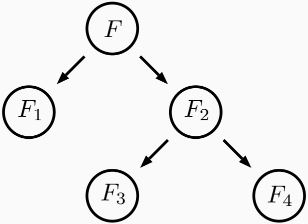
- Partition F into subsets $F_1, \ldots, F_k$, and solve the subproblems $\min c(x)$ s.t. $x \in F_i$. [May be as difficult as the original one, hence split into further subproblems - **branching part**.]
- Compute lower bounds $b(F_i)$ for the subproblems. [A lower bound might be easy to obtain, e.g. LP relaxation - **bounding part**.]
- Maintain an upper bound U on the optimal cost. [E.g. the cost of the best feasible solution thus far.]

Key observation: If $b(F_i) \geqslant U$, then the subproblem needs not to be considered further.

Branch and bound II

Algorithm (general step):

1. Select an active subproblem F_i.
2. If the subproblem is infeasible, delete it; otherwise compute $b(F_i)$.
   - If $b(F_i) \geqslant U$, delete the subproblem.
   - If $b(F_i) < U$, either determine an optimal solution for F_i, or break it into further (active) subproblems.



Remarks:

- Choosing the subproblem, e.g. BFS or DFS.
- Computing the lower bounds, e.g. LP relaxation.
- Breaking into subproblems.

Rounding methods

Given a minimization problem, an α-approximation algorithm provides a solution of value at most $\alpha \cdot OPT$.

Integer program

$$\min c^T \cdot x$$

$$A \cdot x \leqslant b$$

$$x \in \mathbb{Z}^n$$

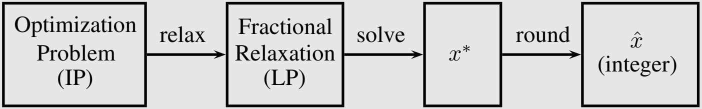
Naive approach:

1. remove the integrality constraint,
2. solve the corresponding LP, and
3. round the entries of the solution to get an integer solution.

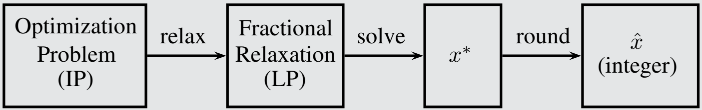
Problems:

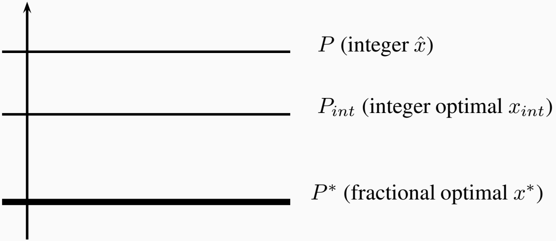
- the solution may not be feasible — Maintain feasibility.
- the solution may not be optimal — Approximation?

Analysing the solution



Analysing the solution





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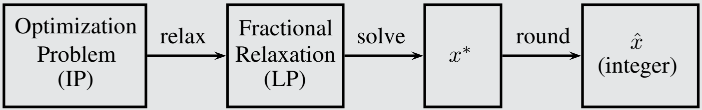
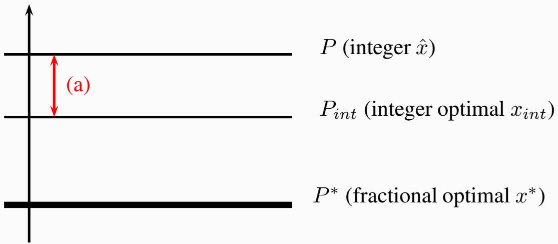
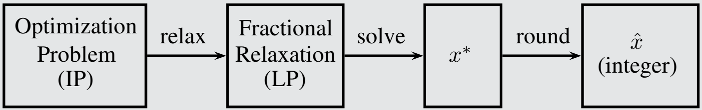
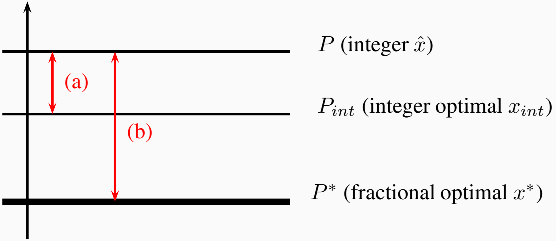




(a) = Approximation ratio between $\hat{x}$ and x_{int}.

Analysing the solution





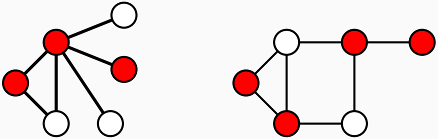
(a) = Approximation ratio between $\hat{x}$ and x_{int}.

(b) = Approximation ration between $\hat{x}$ and x^*.

Vertex cover I

Problem

Find a minimum number of vertices covering every edge of a graph.



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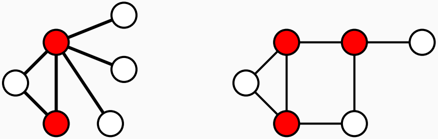

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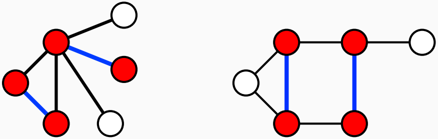

Simple algorithm:

Step 1. Take an inclusionwise maximal matching M.

Step 2. Consider the end vertices of the matching edges.

Observation

This gives a 2-approximation.

- One of Karp's 21 NP-complete problems.
- Moreover, it is APX-complete.
  - No better than 1.3606-approx. unless **P** = **NP**.
  - No better than 2-approx. assuming **UGC**.

Vertex cover II

IP formulation

$$\min \sum_{v \in V} x_v$$

$$x_u + x_v \geqslant 1 \quad \text{for } uv \in E$$

$$x_v \in \{0,1\} \quad \text{for } v \in V$$

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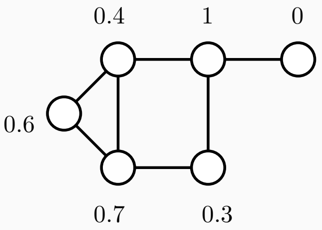
Take a fractional solution x^*.

Step 2.

Define

$$\hat{x}_v = \begin{cases} 1 & \text{if } x_v^* \geqslant 1/2, \\ 0 & \text{otherwise.} \end{cases}$$



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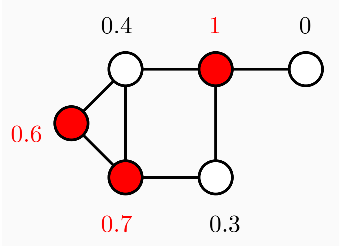

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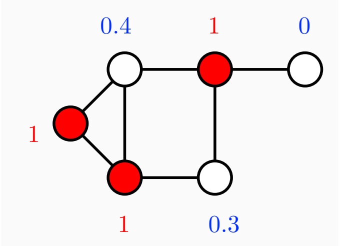

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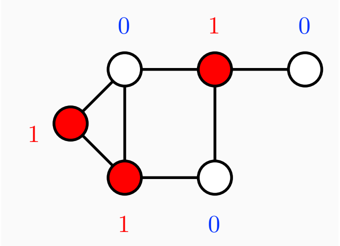

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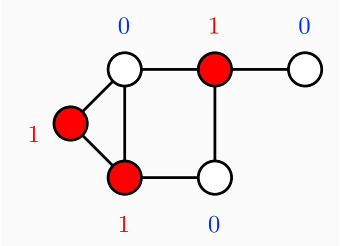

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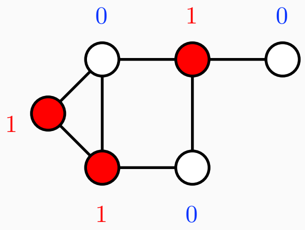

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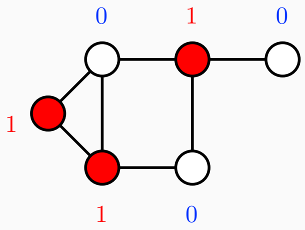

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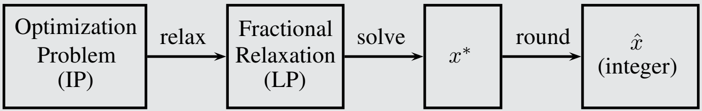
Proof.
Note that $\hat{x}$ is integral, feasible, and $\hat{x}_v \leqslant 2 \cdot x_v^*$. Hence

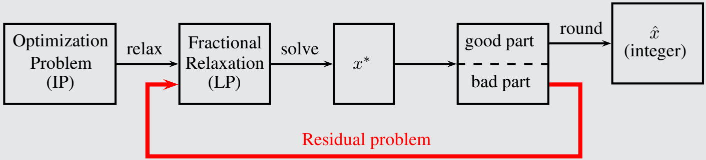
$$\sum_{v \in V} \hat{x}_v \leqslant 2 \cdot \sum_{v \in V} x_v^* \leqslant 2 \cdot OPT.$$

Threshold vs. iterative rounding

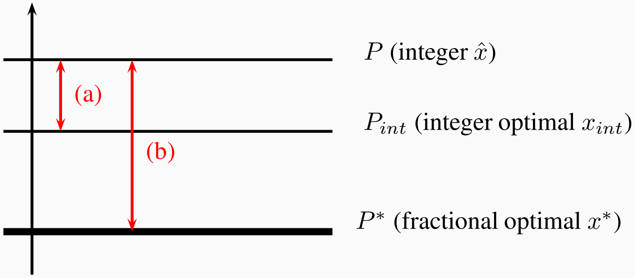
Threshold rounding



Iterative rounding



Integrality gap



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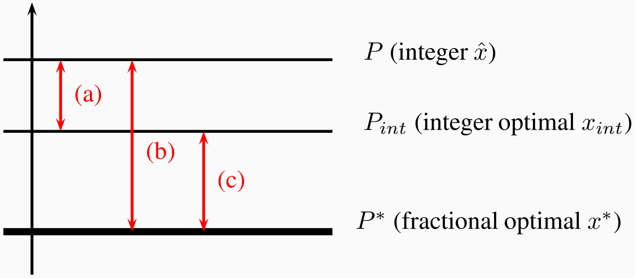

(a) = Approximation ratio between $\hat{x}$ and x_{int}.

(b) = Approximation ration between $\hat{x}$ and x^*.

(c) = Integrality gap.

Heuristics - Local search

$$\min \; c(x)$$
$$\text{s.t. } x \in F$$

Algorithm:

1. Start at some $x \in F$.
2. Evaluate $c(x)$, and evaluate $c(y)$ for "neighbors" $y \in F$ of x.
   - If $c(y) < c(x)$, then move to y and repeat.
   - Otherwise stop: *local optimum* has been found.

Remarks:

- Specifics are determined once "neighbors" are defined.
- Simplex method can be viewed as a special case.
- In practice: run repeatedly starting from different initial solutions.
- Tradeoff: **better solution** is likely to be obtained when considering **larger neighborhood**, but this results in **slower running time**.

Heuristics - Simulated annealing I

Main drawback of local search: Only finds local minimum.

Idea: Allow occasional moves to feasible solutions with higher costs.

Algorithm: For every state $x \in F$, a set $N(x) \subseteq F$ of neighbors is given $(y \in N(x) \Leftrightarrow x \in N(y))$.

1. Start from state $x \in F$.
2. Select a random neighbor y of x with probability q_{xy}.
   [Here $q_{xy} \geqslant 0$ and $\sum_{y \in N(x)} q_{xy} = 1$.]
3. Compute the difference $c(y) - c(x)$.
   - If $c(y) \leqslant c(x)$, then move to state y.
   - If $c(y) > c(x)$, then move to state y with probability $e^{-(c(y)-c(x))/T}$.

Remarks:

- When the temperature T is small - cost increases are unlikely.
- When T is large - the value of $c(y) - c(x)$ has insignificant effect.

Heuristics - Simulated annealing II

The procedure evolves as a Markov chain. Let $A = \sum_{z \in F} e^{-c(z)/T}$.

Steady-state distribution:

$$\pi(x) = \frac{e^{-c(x)/T}}{A},$$

$\Rightarrow \pi(x)$ falls exponentially with $c(x)$. Hence if T is small, then almost all of the steady-state probability is concentrated on states minimizing $c(x)$ **globally**. Should we set T to some very small constant then?

Drawback: the lower the value of T, the harder it is to escape from a local minimum and the longer it takes to reach steady-state.

Instead: Let the temperature vary with time:

$$T(t) = \frac{C}{\log t}.$$

Thm.

If C is sufficiently large, then $\lim_{t \to \infty} P(x(t) \text{ is optimal}) = 1$.