Deep learning and continuous optimization

Submission deadline:

26 February, 14:00

Exercise 1 (1pt)**.** Prove that for an arbitrary function $f\colon \mathbb{R}^n \to \mathbb{R}$, the conjugate function $f^*(y) = \sup\{y^T x - f(x) \mid x \in \mathbb{R}^n\}$ is convex.

Exercise 2 (2pts)**.** Show that $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if $g(t) := f(ty + (1-t)x)$ is convex for any $x, y \in dom(f)$.

Exercise 3 (2pts)**.** Prove that

(a) e^{ax} is convex on $\mathbb{R}$ for any $a \in \mathbb{R}$,

(b) x^a is convex on $\mathbb{R}_{>0}$ when $a \geq 1$ or $a \leq 0$, and is concave when $0 < a < 1$,

(c) $\log(x)$ is concave on $\mathbb{R}_{>0}$,

(d) $x \log(x)$ is convex on $\mathbb{R}_{>0}$.

Exercise 4 (3pts)**.** Let us consider the following functions:

$$\begin{aligned} f_1(w_1, w_2) &= \frac{1}{2}w_1^2 + \frac{7}{2}w_2^2 \\ f_2(w_1, w_2) &= 100(w_2 - w_1^2)^2 + (1 - w_1)^2 \qquad (Rosenbrock's function) \\ f_3(w_1, w_2) &= \frac{1}{2}w_1^2 + w_1 \cos w_2. \end{aligned}$$

(a) Calculate the gradient of the functions.

(b) Determine the global minimum of the functions.

(c) Are these function convex?

Exercise 5 (2pts)**.** Consider the optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}} \quad & x^2 + 2x + 4 \\ s.t. \quad & x^2 - 4x \leq -3 \end{aligned}$$

(a) Solve this problem, i.e., find the optimal solution.

(b) Derive the dual problem $\max_{\lambda \geq 0} g(\lambda)$.