Matroid theory
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Exercise 1. Let $M = (S, \mathcal{I})$ be a matroid whose ground set decomposes into two disjoint bases, and consider a coloring of S such that each color is used at most twice. Show that

(a) S can be covered by three rainbow independent sets of M one of which is a basis.

(b) S can be covered by $\lfloor \log |S| \rfloor + 1$ rainbow bases.

Exercise 2. Let $D = (V, A)$ be a directed graph. Prove that the maximum size of a branching (i.e. a subgraph in which each vertex has indegree at most 1 and contains no cycle in the undirected sense) is equal to $|V|$ minus the number of source-components (i.e. strongly connected components with in-degree zero).

Exercise 3. Let $F_0, \ldots, F_k$ denote the common independent sets generated by the matroid intersection algorithm. For $i = 1, 2$, let σ_i denote the closure operator of M_i. Show that $\sigma_i(f_j) \subseteq \sigma_i(F_{j+1})$ holds for $j = 0, \ldots, k-1$ and $i = 1, 2$.

Exercise 4. Let $M = (S, \mathcal{F})$ be a matroid, $c : S \to \mathbb{R}$ be a weight function, and $k \in \mathbb{Z}_+$. Furthermore, let $\mathcal{F}^k = \{F \in \mathcal{F} \mid |F| = k\}$. Prove that an independent set F of size k is c-maximal in $\mathcal{F}^k$ if and only if

$$c(y) \leq c(x) \text{ for every } y \in S - F, F + y \notin \mathcal{F}, x \in C(F, y), \text{ and}$$
$$c(y) \leq c(x) \text{ for every } y \in S - F, F + y \in \mathcal{F}, x \in F.$$

Exercise 5. Let B be a maximum-weight basis of a matroid $M = (S, r)$ with respect to weight function c. Let $x_1, \ldots, x_k \in B$ and $y_1, \ldots, y_k \in S - B$ such that

$$x_i \in C(B, y_i) \text{ for } i = 1, \ldots, k,$$
$$c(x_i) = c(y_i) \text{ for } i = 1, \ldots, k,$$
$$x_h \notin C(B, y_j) \text{ if } h > j, c(x_h) = c(y_h).$$

Prove that $B' = B - \{x_1, \ldots, x_k\} + \{y_1, \ldots, y_k\}$ is also a maximum weight basis.

Exercise 6. Given two matroids $M_1 = (S, \mathcal{F}_1)$ and $M_2 = (S, \mathcal{F}_2)$, we call a set $F \subseteq S$ a basis-intersection if it can be obtained as the intersection of a basis of M_1 and a basis of M_2. Give an algorithm for deciding if a given common independent set F is a basis-intersection or not.

Exercise 7. Let $\mu_{\min}$ and $\mu_{\max}$ denote the minimum and the maximum size of a basis-intersection of M_1 and M_2. Prove that for every $\mu_{\min} \leq j \leq \mu_{\max}$ there exists a basis-intersection of size j.

Exercise 8 (Brualdi)**.** Let $G = (S, T; E)$ be a bipartite graph, and $M_1 = (S, r_1)$ and $M_2 = (T, r_2)$ be matroids. We call a matching $F \subseteq E$ strongly independent if it covers independent sets both in M_1 and M_2. Prove that the maximum size of a strongly independent matching is equal to

$$\min\{r_1(X) + r_2(Y) \mid X \subseteq S, Y \subseteq T, X \cup Y \text{ covers every edge of } G\}.$$

Exercise 9 (Kundu and Lawler)**.** Let $M_1 = (S, r_1)$ and $M_2 = (S, r_2)$ be matroids with closure operators σ_1 and σ_2, respectively. Let F_1 and F_2 be two common independent sets. Prove that there exists a common independent set F such that $F_1 \subseteq \sigma_1(F)$ and $F_2 \subseteq \sigma_2(F)$.