Matroid theory
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Exercise 1. Let $M_1 = (S, r_1)$ and $M_2 = (S, r_2)$ be two matroids on the same ground set. Show that the problem of finding a common basis of the two matroids can be reduced to the case when one of the matroids is a partition matroid with upper bound one on every partition class.

Given two matroids $M_1 = (S, r_1)$ and $M_2 = (S, r_2)$, the **covering number** $\beta(M_1, M_2)$ **of their intersection** is the minimum number of common independent sets needed to cover S.

Exercise 2. Prove that $\beta(M_1, M_2) \leq \beta(M_1) \cdot \beta(M_2)$.

Aharoni and Berger conjectured a much stronger upper bound.

Conjecture 1 (Aharoni and Berger). *$\beta(M_1, M_2) = \max\{\beta(M_1), \beta(M_2)\}$ if $\beta(M_1) \neq \beta(M_2)$ and $\beta(M_1, M_2) \leq \max\{\beta(M_1), \beta(M_2)\} + 1$ otherwise.*

Exercise 3. Prove that if both M_1 and M_2 are partition matroids, then $\beta(M_1, M_2) = \max\{\beta(M_1), \beta(M_2)\}$.

Exercise 4. Prove that if both M_1 and M_2 are strongly base orderable, then $\beta(M_1, M_2) = \max\{\beta(M_1), \beta(M_2)\}$.

Exercise 5. Let M_1 and M_2 be k-coverable rank-r matroids on a common ground set of size $k \cdot r$. Prove that M_1 and M_2 have a common basis.

Exercise 6. Let $M = (S, \mathcal{I})$ be a matroid whose ground set decomposes into two disjoint bases, and consider a coloring of S such that each color is used at most twice. Show that

(a) S can be covered by three rainbow independent sets of M one of which is a basis.

(b) S can be covered by $\lfloor \log |S| \rfloor + 1$ rainbow bases.

Given a graph $G = (V, E)$, a **proper edge coloring** of G is an assignment of colors to the edges so that no two adjacent edges have the same color. The **edge coloring number** is the smallest integer k for which G has a proper edge coloring by k colors. The classical result of Kőnig states that the edge coloring number of bipartite graphs is equal to its maximum degree. If a list L_e of colors is given for each edge $e \in E$, then a **proper list edge coloring** of G is a proper edge coloring such that every edge e receives a color from its list L_e. The **list edge coloring number** is the smallest integer k for which G has a proper list edge coloring whenever $|L_e| \geq k$ for every $e \in E$. Galvin showed the following.

Theorem 2 (Galvin). *The list edge coloring number of a bipartite graph is equal to its edge coloring number, that is, to its maximum degree.*

We can extend these notions to matroids as well. A **coloring** of the ground set of a matroid M is called **proper** if each color class form an independent set of M. The **coloring number** of M is the minimum number of colors in a proper coloring. Note that this exactly the same as the covering number $\beta(M)$. If a list L_s of colors is given for each element $s \in S$, then a **list coloring** of M is a coloring of the ground set such that every element s receives a color from its list L_s, and elements having the same color form independent sets of M. The **list coloring number** $\beta_\ell(M)$ is the smallest integer k for which M has a proper list coloring whenever $|L_s| \geq k$ for every $s \in S$. The coloring number $\beta(M_1 \cap M_2)$ and the list coloring number $\beta_\ell(M_1 \cap M_2)$ can be defined analogously for the intersection of two matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ on the same ground set S.

Exercise 7. Prove that if both M_1 and M_2 are of rank 2, then $\beta_\ell(M_1 \cap M_2) = \beta(M_1 \cap M_2)$.

Exercise 8. Prove that if both M_1 and M_2 are transversal matroids, then $\beta_\ell(M_1 \cap M_2) = \beta(M_1 \cap M_2)$.

Exercise 9. Prove that if both M_1 and M_2 are graphic matroids, then $\beta_\ell(M_1 \cap M_2) \leq 2 \cdot \beta(M_1 \cap M_2)$.