Matroid theory
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Exercise 1. Let $M = (S, r)$ be a rank-r transversal matroid on S. Prove that there exists a bipartite graph $G = (S, T; E)$ implying M such that $|T| = r(S)$.

Conjecture 2 (Rota's basis conjecture)**.** *Let M be a matroid of rank n whose ground set is partitioned into n disjoint bases $B_1, \ldots, B_n$. Then there exist n pairwise disjoint transversal bases, where a basis is **transversal** if it intersects B_i for $i = 1, \ldots, n$.*

Exercise 3. Let $B_1 \in B(M_1)$, $B_2 \in B(M_2)$ be disjoint bases of rank-n paving matroids on the same ground set, where $n \geq 3$. Let X be a two-element subset of B_1. Then there is some $x \in X$, $y \in B_2$ such that $(B_1 - x) \cup y \in B(M_1)$ and $(B_2 - y) \cup x \in B(M_2)$.

Exercise 4. Let $B_1, \ldots, B_n$ be disjoint sets of size $n \geq 3$, and let $M_1 \ldots, M_n$ be rank-n paving matroids on $B_1 \cup \cdots \cup B_n$ such that B_i is a basis of M_i for each $i = 1, \ldots, n$. Then there is an ordering of the elements of B_1 as $a_1, \ldots, a_n$ and a transversal $\{b_2, \ldots, b_n\}$ of $(B_2, \ldots, B_n)$ such that for all $j = 2, \ldots, n$ the set $(B_1 - \{a_2, \ldots, a_j\}) \cup \{b_2, \ldots, b_j\}$ is a basis of M_1, and $(B_j - b_j) \cup a_j$ is a basis of M_j.

Exercise 5. It is known that if S has size 9 and it decomposes into B_1, B_2 and B_3 where B_i is the basis of a paving matroid M_i of rank 3, then it decomposes into three transversals B'_1, B'_2 and B'_3 where B'_i is a basis of M_i. Using this and the previous two exercises, verify Rota's basis conjecture for paving matroids.

The **covering number** of a matroid M, denoted by $\beta(M)$, is the minimum number of independent sets needed to cover its ground set. Given matroids $M = (S, \mathcal{I})$ and $N = (S, \mathcal{J})$, we say that N is a **reduction** of M if $\mathcal{J} \subseteq \mathcal{I}$, that is, every independent set of N is independent in M as well. In notation, we will denote N being a reduction of M by $N \preceq M$. For the current set of exercises, a **partition matroid** is a matroid $N = (S, \mathcal{J})$ such that $\mathcal{J} = \{X \subseteq S : \ |X \cap S_i| \leq 1 \text{ for } i = 1, \ldots, q\}$ for some partition $S = S_1 \cup \cdots \cup S_q$. Clearly, the covering number of N is $\beta(N) = \max\{|S_i| : \ i = 1, \ldots, q\}$.

Exercise 6. Let $M = (S, \mathcal{I})$ be a k-coverable graphic matroid. Prove that there exists a $(2k-1)$-coverable partition matroid N with $N \preceq M$, and the bound for the covering number of N is tight.

Exercise 7. Let $M = (S, \mathcal{I})$ be a k-coverable transversal matroid. Prove that there exists a k-coverable partition matroid N with $N \preceq M$.

Given a matroid together with a coloring of its ground set, a subset of its elements is called **rainbow colored** if it does not contain two elements of the same color. Accordingly, a coloring is called **rainbow circuit-free** if no circuit or cut is rainbow colored. It is not difficult to check that there is a one-to-one correspondence between reductions of M to partition matroids and rainbow circuit-free colorings of M.

Exercise 8. Every loopless matroid of rank r has a rainbow circuit-free coloring with exactly r colors.

Exercise 9. Characterize those graphs $G = (V, E)$ for which E is the union of two disjoint spanning trees, and G has a rainbow cycle-free coloring with exactly $|V| - 1$ colors using each color twice.

Exercise 10. Let $G = (V, E)$ be a graph on n vertices. Prove that if E is colored with exactly $n - 1$ colors, then G either contains a rainbow cycle or a monochromatic cut.