Matroid theory
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Exercise 1. Let S be a ground set, $r \in \mathbb{Z}_+$ be a non-negative integer, and $\mathcal{B} \subseteq 2^S$ be a family of sets satisfying the following properties:

(B1') $\mathcal{B} \neq \emptyset$,

(B2') $|B| = r$ for each $B \in \mathcal{B}$,

(B3') for distinct $A, B \in \mathcal{B}$ there exist $a \in A - B$ and $b \in B - A$ such that $A - a + b \in \mathcal{B}$ and $B + a - b \in \mathcal{B}$.

Prove that $\mathcal{B}$ forms the family of bases of a matroid.

Exercise 2. Let $M = (S, \mathcal{F})$ be a matroid and $c \in \mathbb{R}^S$ be a weight function. Give an algorithm for deciding if a given independent set I can be extended to a maximum weight basis of M.

Exercise 3. Let $M = (S, \mathcal{F})$ be a matroid. Give an algorithm for deciding if there exists a basis that has maximum weight with respect to all of the weight functions $c_1, \ldots, c_k$.

Exercise 4. Let $G = (X, Y; E)$ be a bipartite graph having a unique perfect matching. Prove that there exist orderings $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ of the vertices of X and Y such that $x_i y_i \in E$ for $1 \leq i \leq n$, and $x_i y_j \notin E$ for $1 \leq i < j \leq n$.

Exercise 5. Let $G = (V, E)$ be a connected undirected graph.

(a) Find a minimum sized subset $F \subseteq E$ intersecting every spanning tree of G.

(b) Let $c\colon E \to \mathbb{R}$ be a cost function. Find a minimum cost subset $F \subseteq E$ intersecting every spanning tree of G.

(c) Let $w\colon E \to \mathbb{R}$ be a weight function. Find a minimum sized subset $F \subseteq E$ intersecting every maximum weight spanning tree of G.

(d) Let $c\colon E \to \mathbb{R}$ and $w\colon E \to \mathbb{R}$ be a cost and a weight functiton. Find a minimum cost subset $F \subseteq E$ intersecting every maximum weight spanning tree of G.

Exercise 6. Let $M = (S, \mathcal{B})$ be a matroid and $w : S \to \mathbb{R}$ be a weight function. Decide if there exists a basis $B \in \mathcal{B}$ with non-integral total weight.

Exercise 7. Let $M = (S, \mathcal{B})$ be a matroid. Decide if there exists a weight function $w : S \to \mathbb{R}$ such that $w(S)$ is non-integral but $w(B)$ is integral for every basis $B \in \mathcal{B}$.

Exercise 8. Prove that the following problem is hard: Given a matroid together with a weight function and a real number C, find a basis of weight exactly C.

Exercise 9. Prove that a matroid $M = (S, r)$ is connected if and only if its dual $M^* = (S, r^*)$ is connected.

Exercise 10. A matroid M is **strongly base orderable** if for any two bases A, B there exists a bijection $\varphi : A \to B$ such that

$$A - X + \varphi(X) \text{ is a basis for every } X \subseteq A. \qquad \text{(SBO)}$$

Let A and B be disjoint spanning trees of the same simple undirected graph G. Prove that there is no bijection between A and B satisfying (SBO).

Exercise 11. Let $G = (V, E)$ be an undirected graph with $|V| = n$ such that E can be decomposed into two disjoint spanning trees A and B. Prove that there exists a bijection $\varphi : A \cup B \to \{1, \ldots, 2n - 2\}$ for which every cycle of G contains two consecutive numbers.