Matroid theory
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Exercise 1. Prove that indipendence axioms $(I3)$ and $(I3')$ are equivalent.

Exercise 2. Prove the sets of axioms $\{(I1),(I2),(I3)\}$ and $\{(I1),(I2),(I3'')\}$ are equivalent.

Exercise 3. Prove the sets of axioms $\{(I1),(I2),(I3')\}$ and $\{(I1),(I2),(I3''')\}$ are equivalent.

Exercise 4. Let $b\colon 2^S \to \mathbb{R}$ be a set function. Prove that b is submodular if and only if $b(X+s)-b(X) \geq b(Y+s)-b(Y)$ holds for every $X \subseteq Y \subseteq S$, $s \in S-Y$.

Exercise 5. Let $C_1,\ldots,C_k$ be pairwise disjoint circuits in a matroid M, and let $x_i \in C_i$ for $i=1,\ldots,k$. Furthermore, let C be a circuit of M distinct from all C_is. Verify that M has a circuit that is disjoint from $\{x_1,\ldots,x_k\}$.

Exercise 6. Give an example showing that the following stronger variant of the circuit axiom does not hold: for every pair C_1, C_2 of circuits, $f \in C_1 \cap C_2$, $e_1 \in C_1 - C_2$ and $e_2 \in C_2 - C_1$ there exists a circuit $C \in C_1 \cup C_2 - f$ containing both e_1 and e_2.

Exercise 7. Let $M = (S, \mathcal{B})$ be a matroid and $c\colon 2^S \to \mathbb{R}$ be a weight function. Prove that the family of maximum weight bases satisfies the basis axioms.

Exercise 8. Let $M = (S, \mathcal{B})$ be a matroid and $c\colon 2^S \to \mathbb{R}$ be a weight function. Prove that every maximum weight basis can be obtained by the greedy algorithm.

Exercise 9. Let C and K be a circuit and a cut of the same matroid. Prove that $|C \cap K| \neq 1$.

Exercise 10. Prove that hyperplanes of a matroid M are exactly the complements of its cuts.

Exercise 11. Prove that the circuits of a matroid M are exactly the cuts of its dual M^*.

Exercise 12. Let $G = (V, E)$ be an undirected graph. Prove that for any $X, Y \subseteq V$ we have $c(X) + c(Y) \leq c(X \cap Y) + c(X \cup Y) + d(X, Y)$, where $c(Z)$ denotes the number of components after deleting Z, while $d(X, Y)$ denotes the number of edges going between $X - Y$ and $Y - X$.

Exercise 13. A matroid M is called binary if it can be represented over the field $GF(2)$, that is, there exists a 0-1 matrix A whose columns are identified with the elements of the matorid in such a way that a subset of columns of A is independent over $GF(2)$ if and only if the corresponding elements form an independent set of M. Verify that the graphic matroid of any graph is a binary matroid.