Problem set 9
List colorings, Kano's conjectures, Sperner's lemma

Given a graph $G = (V, E)$, a **proper edge coloring** of G is an assignment of colors to the edges so that no two adjacent edges have the same color. The **edge coloring number** is the smallest integer k for which G has a proper edge coloring by k colors. The classical result of Kőnig states that the edge coloring number of bipartite graphs is equal to its maximum degree. If a list L_e of colors is given for each edge $e \in E$, then a **proper list edge coloring** of G is a proper edge coloring such that every edge e receives a color from its list L_e. The **list edge coloring number** is the smallest integer k for which G has a proper list edge coloring whenever $|L_e| \geq k$ for every $e \in E$. Galvin showed the following.

Theorem 1 (Galvin)**.** *The list edge coloring number of a bipartite graph is equal to its edge coloring number, that is, to its maximum degree.*

We can extend these notions to matroids as well. A **coloring** of the ground set of a matroid M is called **proper** if each color class form an independent set of M. The **coloring number** of M is the minimum number of colors in a proper coloring. Note that this exactly the same as the covering number $\beta(M)$. If a list L_s of colors is given for each element $s \in S$, then a **list coloring** of M is a coloring of the ground set such that every element s receives a color from its list L_s, and elements having the same color form independent sets of M. The **list coloring number** $\beta_\ell(M)$ is the smallest integer k for which M has a proper list coloring whenever $|L_s| \geq k$ for every $s \in S$. The coloring number $\beta(M_1 \cap M_2)$ and the list coloring number $\beta_\ell(M_1 \cap M_2)$ can be defined analogously for the intersection of two matroids $M_1 = (S, \mathscr{I}_1)$ and $M_2 = (S, \mathscr{I}_2)$ on the same ground set S.

Problem 1. Prove that if both M_1 and M_2 are of rank 2, then $\beta_\ell(M_1 \cap M_2) = \beta(M_1 \cap M_2)$.

Problem 2. Prove that if both M_1 and M_2 are transversal matroids, then $\beta_\ell(M_1 \cap M_2) = \beta(M_1 \cap M_2)$.

Problem 3. Prove that if both M_1 and M_2 are graphic matroids, then $\beta_\ell(M_1 \cap M_2) \leq 2 \cdot \beta(M_1 \cap M_2)$.

Problem 4. Prove that if $D = (V, A)$ is the disjoint union of two spanning r-arborescences and each arc has a list of size 2, then D has a proper list coloring in the sense that each arc receives a color from its list and the sets of identically colored arcs form branchings.

Problem 5. Prove that if a directed graph $D = (V, A)$ is the union of k arborescences and each arc has a list of at least $\lceil 3k/2 \rceil$ colors, then D has a proper list coloring as in Problem 4.

Open problem 6. Can we improve the bound in Problem 5? E.g., having lists of size 4 suffices if $k = 3$?

Open problem 7. Prove that if $D = (V, A)$ is the disjoint union of k spanning r-arborescences and each arc has a list of size k, then D has a proper list coloring.

Motivated by conjectures of Kano, Lemos proposed the following conjecture.

Conjecture 2. *Let $M = (S, \mathscr{B})$ be B_1 and B_2 be two disjoint bases. Furthermore, let $w : S \to \mathbb{R}$ be a weight function such that $w(B_1) > w(B_2)$ and B_1 is the unique basis with its weight. Then M has r bases all having different weights less than $w(B_1)$.*

Problem 8. Verify Conjecture 2 when B_2 is a minimum weight basis.

Problem 9. Verify Conjecture 2 for strongly base orderable matroids.

Problem 10. Verify Conjecture 2 if $w(s) \in \{0, 1\}$ for each $s \in S$.

Given a coloring of the vertices of a polytope by n colors, a facet is called **multicolored** if it contains vertices of each color.

Theorem 3 (Polyhedral Sperner's lemma)**.** *Let P be an n-dimensional polytope. Suppose we have a coloring of the vertices of P by n colors for which there exists a multicolored facet. Then there exists another multicolored facet.*

Problem 11. Instead of a coloring, consider a mapping φ from the ground set of a matroid M to the vertices of P. Verify the following: if the elements mapped to some facet of P form a basis of M, then there exists another facet for which the same holds.

Problem 12. Let $G = (V, E)$ be an undirected graph, $w : E \to \mathbb{R}$ be a weight function and $c : E \to \mathbb{R}$ be a cost function. Find a minimum cost subset of edges whose deletion decreases the maximum weight of a spanning tree.

Problem 13. Let $G = (S, T; E)$ be a matching covered (i.e., every edge is contained in some perfect matching) connected bipartite graph and $B \subseteq E$. Then $|M \cap B|$ is even for every perfect matching M of G if and only if B is the mod 2 sum of an even number of stars.