Problem set 8
Weights, matroid intersection, and matroid union

The **sum** $M_1 + M_2$ of $M_1 = (S, \mathscr{I}_1)$ and $M_2 = (S, \mathscr{I}_2)$ on the same ground set is a matroid $M = (S, \mathscr{I})$ whose independent sets are the disjoint unions of an independent set of M_1 and an independent set of M_2. For a weight function $w\colon S \to \mathbb{R}$ and a subset $X \subseteq S$, define $w(X) = \sum_{e \in X} w(e)$. For a family $\mathscr{F} \subseteq 2^S$, a subset $X \subseteq S$ is w**-maximal in** $\mathscr{F}$ if $X \in \arg\max\{w(Y) \mid Y \in \mathscr{F}\}$. We define $\mathscr{I}_i^k = \{X \in \mathscr{I}_i \mid |X| = k\}$ for $i = 1, 2$ and $k = 0, 1, \ldots, n$. Frank's weight-splitting theorem provides a weighted counterpart of Edmonds' theorem.

Theorem 1 (Frank)**.** *For any weight function $w\colon S \to \mathbb{R}$ and any k with $\mathscr{I}_1^k \cap \mathscr{I}_2^k \neq \emptyset$, the following equation holds:*

$$\max\left\{ w(I) \;\middle|\; I \in \mathscr{I}_1^k \cap \mathscr{I}_2^k \right\} = \min\left\{ \max_{I_1 \in \mathscr{I}_1^k} w_1(I_1) + \max_{I_2 \in \mathscr{I}_2^k} w_2(I_2) \;\middle|\; w_1 + w_2 = w \right\}.$$

Problem 1. Let $M_1 = (S, \mathscr{B}_1)$ and $M_2 = (S, \mathscr{B}_2)$ be matroids and let $X \subseteq S$. Find $B_1 \in \mathscr{B}_1$ and $B_2 \in \mathscr{B}_2$ such that $B_1 \cap B_2 \subseteq X$.

Problem 2. Let $M = (S, \mathscr{B})$ be a matroid and $w : S \to \mathbb{R}$ be a weight function. Decide if there exists a basis $B \in \mathscr{B}$ with non-integral total weight.

Problem 3. Let $M = (S, \mathscr{B})$ be a matroid. Decide if there exists a weight function $w : S \to \mathbb{R}$ such that $w(S)$ is non-integral but $w(B)$ is integral for every basis $B \in \mathscr{B}$.

Open problem 4. Let $M_1 = (S, \mathscr{B}_1)$, $M_2 = (S, \mathscr{B}_2)$ be matroids and $w : S \to \mathbb{R}$ be a weight function. Decide if there exists a common basis with non-integral total weight.

Problem 5. Prove that the following problem is hard: Given a matroid together with a weight function and a real number C, find a basis of weight exactly C.

Problem 6. Let M be a matroid, B_0 be a basis, $w : S \to \mathbb{R}$ a weight function, and C be real number. Find a basis B of M with $w(B) \leq C$ such that $|B \cap B_0|$ is as large as possible.

Problem 7. Given a matroid $M = (S, \mathscr{I})$ and an integer $k > 0$, find a partition of S into k parts such that the union of any $k-1$ of them is independent.

Open problem 8. Given a matroid $M = (S, \mathscr{I})$ and integers $k \geq \ell > 0$, find a partition of S into k parts such that the union of any ℓ of them is independent.