Problem set 7
Rota's basis conjecture

Problem 1. Exercises from earlier weeks:

- Problem set 5, Problem 1. **Hint:** Consider the two cases when each r_i is at least $r/2$ and when there exists an r_i less than $r/2$.
- Problem set 5, Problem 3. **Hint:** Use the direct sum of M_2 and the matroid obtained by taking $\beta(M_1)-1$ parallel copies of each element in the dual of M_1.
- Problem set 5, Problem 7. **Hint:** Use the symmetric exchange property of matroids, due to Greene: if B_1 and B_2 are bases of the same matroid M and $X \subseteq B_1$, then there exists $Y \subseteq B_2$ such that both $B_1 - X + Y$ and $B_2 + X - Y$ are bases.
- Problem set 6, Problem 5. **Hint:** Build up an arborescence in such a way that there always exists a color whose root vertex is reached but the color itself was not used in the arborescence.

Conjecture 1 (Rota's basis conjecture)**.** *Let M be a matroid of rank n whose ground set is partitioned into n disjoint bases $B_1,\dots,B_n$. Then there exist n pairwise disjoint transversal bases, where a basis is* ***transversal*** *if it intersects B_i for $i=1,\dots,n$.*

Problem 2. Prove that Conjecture 1 holds for strongly base orderable matroids.

Problem 3. Let $B_1 \in B(M_1)$, $B_2 \in B(M_2)$ be disjoint bases of rank-n paving matroids on the same ground set, where $n \geq 3$. Let X be a two-element subset of B_1. Then there is some $x \in X$, $y \in B_2$ such that $(B_1 - x) \cup y \in B(M_1)$ and $(B_2 - y) \cup x \in B(M_2)$.

Problem 4. Let $B_1,\dots,B_n$ be disjoint sets of size $n \geq 3$, and let $M_1 \dots, M_n$ be rank-n paving matroids on $B_1 \cup \cdots \cup B_n$ such that B_i is a basis of M_i for each $i = 1,\dots,n$. Then there is an ordering of the elements of B_1 as $a_1,\dots,a_n$ and a transversal $\{b_2,\dots,b_n\}$ of $(B_2,\dots,B_n)$ such that for all $j = 2,\dots,n$ the set $(B_1 - \{a_2,\dots,a_j\}) \cup \{b_2,\dots,b_j\}$ is a basis of M_1, and $(B_j - b_j) \cup a_j$ is a basis of M_j.

Problem 5. It is known that if S has size 9 and it decomposes into B_1, B_2 and B_3 where B_i is the basis of a paving matroid M_i of rank 3, then it decomposes into three transversals B'_1, B'_2 and B'_3 where B'_i is a basis of M_i. Using this and the previous two exercises, verify Rota's basis conjecture for paving matroids.

Problem 6. Prove Rota's conjecture for graphic matroids when each B_i is a star.