Problem set 6
Rainbow paths, trees, and arborescences

Problem 1. Let $G = (V, E)$ be a graph that is the union of $n-1$ spanning trees $T_1, \dots, T_{n-1}$. Show that there exists a spanning tree T such that $|T \cap T_i| = 1$ for $i = 1, \dots, n-1$.

The following conjecture proposes an extension of Problem 1 to arborescences.

Conjecture 1. *Let $D = (V, A)$ be a digraph that is the union of $n-1$ spanning arborescences $F_1, \dots, F_{n-1}$. Show that there exists a spanning arborescence F such that $|F \cap F_i| = 1$ for $i = 1, \dots, n-1$.*

By considering $F_1, \dots, F_{n-1}$ to be color classes, we will call such an F to be a **rainbow arborescence**.

Problem 2. Prove that Conjecture 1 holds when each F_i is rooted at the same root r.

Problem 3. Prove that Conjecture 1 holds when each F_i is a star.

Problem 4. Prove that Conjecture 1 holds when each F_i is rooted at either r' or r'' for some $r', r'' \in V$.

Problem 5. Prove that Conjecture 1 holds when the roots of the F_is are pairwise distinct.

Given an edge colored digraph $D = (V, A)$ and $s, t \in V$, it is NP-complete to decide whether D contains a rainbow s-t path.

Problem 6. Prove that under the assumption of Conjecture 1, it is NP-complete to decide if D has a rainbow arborescence F rooted at a fixed vertex r.

Problem 7. Let $D = (V, A)$ be an edge colored digraph in which each color appears at most twice, and let $s, t \in V$. Prove that it is NP-complete to decide whether D contains a rainbow s-t path.

Open problem 8. Let $D = (V, A)$ be a digraph and let $s, t \in V$. Can we decide if D contains a non-shortest s-t path?

Open problem 9. Let $D = (V, A)$ be an edge colored digraph and let $s, t \in V$. Can we decide if D contains a non-rainbow s-t path?

Problem 10. Show that the problem in Open problem 8 reduces to the problem in Open problem 9.