Problem set 5
Covering by common independent sets

Problem 1. Prove that a k-coverable split matroid is reducible to a $2k$-coverable partition matroid.

Recall the conjecture of Aharoni and Berger.

Conjecture 1 (Aharoni and Berger)**.** *$\beta(M_1,M_2) = \max\{\beta(M_1),\beta(M_2)\}$ if $\beta(M_1) \neq \beta(M_2)$ and $\beta(M_1,M_2) \leq \max\{\beta(M_1),\beta(M_2)\}+1$ otherwise.*

Problem 2. Show that in Conjecture 1 we may assume that $|S| = r_1(S)\cdot\beta(M_1) = r_2(S)\cdot\beta(M_2)$.

Problem 3. Show that if $\beta(M_1) < \beta(M_2)$, then in Conjecture 1 we may assume that M_1 is a partition matroid.

One of the most powerful results in matroid optimization is the matroid intersection theorem of Edmonds.

Theorem 2 (Edmonds)**.** *The maximum size of a common independent set of matroids $M_1 = (S,r_1)$ and $M_2 = (S,r_2)$ is*

$$\min_{X\subseteq S}\{r_1(X)+r_2(S-X)\}.$$

Let $P(M_1)$ and $P(M_2)$ denote the convex hulls of the incidence vectors of bases of M_1 and M_2, respectively; these polytopes are called the **base polytopes** of the matroids. It is known that the intersection of $P(M_1)$ and $P(M_2)$, if it is nonempty, is an integral polytope whose vertices are exactly the common bases of M_1 and M_2.

Problem 4. Let M_1 and M_2 be k-coverable rank-r matroids on a common ground set of size $k\cdot r$. Prove that M_1 and M_2 have a common basis.

Problem 5. Let $M = (S,\mathscr{I})$ be a block matroid and consider a coloring of S such that each color is used at most twice. Show that

(a) S can be covered by three rainbow independent sets of M one of which is a basis.

(b) S can be covered by $\lfloor \log|S| \rfloor + 1$ rainbow bases.

Open problem 6. Does the statement of Problem 5(b) hold if we replace $\lfloor \log|S| \rfloor + 1$ by some constant?

Problem 7. Let M_1 and M_2 be 2-coverable matroids on the same ground set. Show that $\beta(M_1,M_2) \leq 3$.