Problem set 4
Covering number and reductions

The **covering number** of a matroid M, denoted by $\beta(M)$, is the minimum number of independent sets needed to cover its ground set. Accordingly, a matroid is called **k-coverable** if $\beta(M) \leq k$. The matroid partition theorem of Edmonds and Fulkerson implies that $\beta(M) = \max\{\lceil |X|/r(X) \rceil : \emptyset \neq X \subseteq S\}$. Analogously, given two matroids $M_1 = (S, r_1)$ and $M_2 = (S, r_2)$, the **covering number** $\beta(M_1, M_2)$ **of their intersection** is the minimum number of common independent sets needed to cover S.

Problem 1. Prove that $\beta(M_1, M_2) \leq \beta(M_1) \cdot \beta(M_2)$.

Aharoni and Berger conjectured a much stronger upper bound.

Conjecture 1 (Aharoni and Berger)**.** *$\beta(M_1, M_2) = \max\{\beta(M_1), \beta(M_2)\}$ if $\beta(M_1) \neq \beta(M_2)$ and $\beta(M_1, M_2) \leq \max\{\beta(M_1), \beta(M_2)\} + 1$ otherwise.*

Problem 2. Prove that if both M_1 and M_2 are partition matroids, then $\beta(M_1, M_2) = \max\{\beta(M_1), \beta(M_2)\}$.

Problem 3. Prove that if both M_1 and M_2 are strongly base orderable, then we have $\beta(M_1, M_2) = \max\{\beta(M_1), \beta(M_2)\}$.

Given matroids $M = (S, \mathscr{I})$ and $N = (S, \mathscr{J})$, we say that N is a **reduction** of M if $\mathscr{J} \subseteq \mathscr{I}$, that is, every independent set of N is independent in M as well. In notation, we will denote N being a reduction of M by $N \preceq M$. For the current set of exercises, a **partition matroid** is a matroid $N = (S, \mathscr{J})$ such that $\mathscr{J} = \{X \subseteq S : |X \cap S_i| \leq 1 \text{ for } i = 1, \ldots, q\}$ for some partition $S = S_1 \cup \cdots \cup S_q$. Clearly, the covering number of N is $\beta(N) = \max\{|S_i| : i = 1, \ldots, q\}$.

Problem 4. Let $M = (S, \mathscr{I})$ be a k-coverable paving matroid of rank $r \geq 2$. Prove that there exists a $\lceil \frac{rk}{r-1} \rceil$-coverable partition matroid N with $N \preceq M$.

Problem 5. Let $M = (S, \mathscr{I})$ be a k-coverable graphic matroid. Prove that there exists a $(2k-1)$-coverable partition matroid N with $N \preceq M$, and the bound for the colouring number of N is tight.

Problem 6. Let $M = (S, \mathscr{I})$ be a k-coverable transversal matroid. Prove that there exists a k-coverable partition matroid N with $N \preceq M$.

Given a matroid together with a coloring of its ground set, a subset of its elements is called **rainbow colored** if it does not contain two elements of the same color. Accordingly, a coloring is called **rainbow circuit-free** if no circuit or cut is rainbow colored. It is not difficult to check that there is a one-to-one correspondence between reductions of M to partition matroids and rainbow circuit-free colorings of M.

Problem 7. Every loopless matroid of rank r has a rainbow circuit-free coloring with exactly r colors.

Problem 8. Characterize those graphs $G = (V, E)$ for which E is the union of two disjoint spanning trees, and G has a rainbow cycle-free coloring with exactly $|V| - 1$ colors using each color twice.

Problem 9. Let $G = (V, E)$ be a graph on n vertices. Prove that if E is colored with exactly $n - 1$ colors, then G either contains a rainbow cycle or a monochromatic cut.