Problem set 3
Split matroids

A rank-r matroid $M=(S,\mathscr{I})$ is called a **paving matroid** if each circuit has size at least r, or in other words, each set of size at most $r-1$ is independent. The matroid is **sparse paving** if M^* is also paving. For a non-negative integer r, a ground set S of size at least r, and a (possibly empty) family $\mathscr{H}=\{H_1,\dots,H_q\}$ of proper subsets of S such that $|H_i\cap H_j|\leq r-2$ for $1\leq i<j\leq q$, the set system $\mathscr{B}_{\mathscr{H}}=\{X\subseteq S \mid |X|=r,\ X\not\subseteq H_i \text{ for } i=1,\dots,q\}$ forms the set of bases of a paving matroid, and in fact every paving matroid can be obtained in this form. We will refer to this as a **hypergraph representation** of M.

Problem 1. Give an example showing that the dual of a paving matroid is not necessarily paving.

Problem 2. Prove that a paving matroid is sparse paving if and only if it has a hypergraph representation in which each hyperedge has size r.

Problem 3. Let S be a ground set of size at least r, $\mathscr{H}=\{H_1,\dots,H_q\}$ be a (possibly empty) collection of subsets of S, and $r,r_1,\dots,r_q$ be non-negative integers satisfying

(H1) $$|H_i\cap H_j|\leq r_i+r_j-r \text{ for } 1\leq i<j\leq q.$$

(a) Prove that $\mathscr{I}=\{X\subseteq S \mid |X|\leq r,\ |X\cap H_i|\leq r_i \text{ for } 1\leq i\leq q\}$ forms the independent sets of a matroid.

(b) Prove that the rank function of the matroid is $r_M(Z)=\min\left\{r,|Z|,\min_{1\leq i\leq q}\{|Z-H_i|+r_i\}\right\}$.

(c) Show that if

(H2) $$|S-H_i|+r_i\geq r \text{ for } i=1,\dots,q$$

holds, then the rank of the matroid is r.

(d) Prove that the hypergraph in Problem 3 can be chosen in such a way that

(H3) $$r_i\leq r-1 \text{ for } i=1,\dots,q,$$

(H4) $$|H_i|\geq r_i+1 \text{ for } i=1,\dots,q.$$

Matroids that can be obtained as described in Problem 3 are called **elementary split matroids**. A matroid is a **split matroid** if it is a direct sum of a single elementary split matroid and some uniform matroids. We call the representation **non-redundant** if all of (H1)–(H4) hold. A set $F\subseteq S$ is called **H_i-tight** if $|F\cap H_i|=r_i$.

Problem 4. Verify the followings.

(a) The class of elementary split matroids is closed under duality.

(b) The class of elementary split matroids is closed under taking minors.

(c) The class of elementary split matroids is closed under truncation.

Problem 5. Let M be a rank-r elementary split matroid with a non-redundant representation $\mathscr{H}=\{H_1,\dots,H_q\}$ and $r,r_1,\dots,r_q$. Let F be a set of size r.

(a) If F is H_i-tight for some index i then F is a basis of M.

(b) If F is both H_i-tight and H_j-tight for distinct indices i and j then $H_i \cap H_j \subseteq F \subseteq H_i \cup H_j$.

Problem 6. Consider a non-redundant representation $\mathscr{H} = \{H_1, \dots, H_q\}$, $r, r_1, \dots, r_q$ of an elementary split matroid M on ground set S. Then $M|H_i \cong U_{r_i,|H_i|}$ and $M/H_i \cong U_{r-r_i,|S-H_i|}$ for $i = 1, \dots, q$.