Problem set 2
Basis-pairs

While studying the structure of symmetric exchanges, Gabow formulated the following conjecture on the so-called **sequential symmetric exchange property**.

Conjecture 4 (Gabow)**.** *Let A and B bases of the same matroid. Then there are orderings $A = (a_1,\dots,a_r)$ and $B = (b_1,\dots,b_r)$ such that $\{a_1,\dots,a_i,b_{i+1},\dots,b_r\}$ and $\{b_1,\dots,b_i,a_{i+1},\dots,a_r\}$ are bases for $i = 0,\dots,r$.*

Problem 1. Verify Gabow's conjecture for strongly base orderable matroids.

Problem 2. Verify Gabow's conjecture for graphic matroids.

Let $\mathscr{X} = (X_1,\dots,X_m)$ be a sequence of bases of a matroid M, and assume that there exist $e \in X_i - X_j$, $f \in X_j - X_i$ with $1 \le i < j \le m$ such that both $X_i - e + f$ and $X_j + e - f$ are bases. Then we say that $\mathscr{X}' = (X_1,\dots,X_{i-1},X_i - e + f,X_{i+1},\dots,X_{j-1},X_j + e - f,X_{j+1},\dots,X_m)$ is obtained from $\mathscr{X}$ by a symmetric exchange. Two sequences $\mathscr{X}$ and $\mathscr{Y}$ **equivalent** if $\mathscr{Y}$ can be obtained from $\mathscr{X}$ by a composition of symmetric exchanges. Furthermore, $\mathscr{X}$ and $\mathscr{Y}$ are called **compatible** if $|\{i:\ s \in X_i, 1 \le i \le m\}| = |\{i:\ s \in Y_i, 1 \le i \le m\}|$ for every $s \in S$. White conjectured the following.

Conjecture 5 (White)**.** *Two basis sequences $\mathscr{X}$ and $\mathscr{Y}$ of the same length are equivalent if and only if they are compatible.*

Problem 3. Verify White's conjecture for $m = 2$ when M is a strongly base orderable matroids. Furthermore, show that at most r exchange steps always suffice.

Problem 4. Verify White's conjecture for $m = 2$ when M is a graphic matroids.

The vertices of the **basis pair graph** $G(M)$ of a matroid M are the ordered triples $\mathbf{A} = (A_1,A_2,A_3)$ where A_1 and A_2 are disjoint bases of M and $A_3 = S - (A_1 \cup A_2)$. Two vertices $\mathbf{A} = (A_1,A_2,A_3)$ and $\mathbf{B} = (B_1,B_2,B_3)$ are adjacent if $\mathbf{B}$ can be obtained from $\mathbf{A}$ by exchanging some pair of elements in two different sets in $\mathbf{A}$, that is, $|A_1 - B_1| + |A_2 - B_2| + |A_3 - B_3| = 2$. Farber conjectured that any ordered triple can be obtained from another through exchanges.

Conjecture 6 (Farber)**.** *The basis pair graph of any matroid is connected.*

Problem 5. Show that if White's conjecture hold for a minor-closed class of matroids, then Farber's conjecture also holds for the same class.

A matroid $M = (S,\mathscr{B})$ is called **equitable** if for any set $X \subseteq S$, there exists a partition $S = B_1 \cup \dots \cup B_k$ of the ground set into pairwise disjoint bases such that $\lfloor |X|/k \rfloor \le |B_i \cap X| \le \lceil |X|/k \rceil$.

Conjecture 7 (Equitability of matroids)**.** *If the ground set of a matroid M can be partitioned into k bases, then M is equitable.*

Problem 6. Prove that it suffices to verify Conjecture 7 for $k = 2$, as then the statement follows for larger values of k.

Problem 7. Prove that both Conjectures 4 and 5 imply Conjecture 7.

Problem 8. Verify Conjecture 7 for graphic matroids.