# Problem set 1
# Basis exchanges

The basis exchange axiom states that for any pair $A,B$ of bases and for any $a \in A-B$ there exists $b \in B-A$ such that $A-a+b$ is a basis as well. The exchange property in fact implies that there exists a bijection $\varphi: A \to B$ such that $A-a+\varphi(a)$ is a basis for each $a \in A$. A major weakness of this result is that it only holds locally: $B+a-\varphi(a)$ might not be a basis for some $a \in A$ as the roles of $A$ and $B$ are not symmetric, and only single exchanges are possible as $A-\{a_1,a_2\}+\{\varphi(a_1),\varphi(a_2)\}$ might not be a basis for some pair $a_1 \neq a_2$.

**Problem 1.** Let $S$ be a ground set, $r \in \mathbb{Z}_+$ be a non-negative integer, and $\mathscr{B} \subseteq 2^S$ be a family of sets satisfying the following properties:
(B1') $\mathscr{B} \neq \emptyset$,
(B2') $|B| = r$ for each $B \in \mathscr{B}$,
(B3') for distinct $A,B \in \mathscr{B}$ there exist $a \in A-B$ and $b \in B-A$ such that $A-a+b \in \mathscr{B}$ and $B+a-b \in \mathscr{B}$.
Prove that $\mathscr{B}$ forms the family of bases of a matroid.

A matroid $M$ is **strongly base orderable** if for any two bases $A,B$ there exists a bijection $\varphi: A \to B$ such that

(SBO) $$A-X+\varphi(X) \text{ is a basis for every } X \subseteq A.$$

Note that this implies $B-\varphi(X)+X$ being a basis as well. In other words, for any pair $A,B$ of disjoint bases of a strongly base orderable matroid $M$, there exists a graph $G$ consisting of a matching between the elements of $A$ and $B$ such that $G$ covers every circuit of $M$ that lie in $A \cup B$. Here **covering** means that every circuit of $M$ spans at least one edge of $G$.

**Problem 2.** Let $A$ and $B$ be disjoint spanning trees of the same undirected graph $G$. Prove that there is no bijection between $A$ and $B$ satisfying (SBO).

**Problem 3.** Let $G=(V,E)$ be an undirected graph with $|V|=n$ such that $E$ can be decomposed into two disjoint spanning trees $A$ and $B$. Prove that there exists a bijection $\varphi: A \cup B \to \{1,\dots,2n-2\}$ for which every cycle of $G$ contains two consecutive numbers.

**Open problem 4.** Let $G=(V,E)$ be an undirected graph with $|V|=n$ such that $E$ can be decomposed into two disjoint spanning trees $A$ and $B$. Prove that there exists a bijection $\varphi: A \cup B \to \{1,\dots,2n-2\}$ for which every cut of $G$ contains two consecutive numbers.

**Problem 5.** Let $M=(S,\mathscr{B})$ be a paving matroid such that $S$ can be decomposed into two disjoint bases $A$ and $B$. Prove that there exists an alternating path between $A$ and $B$ that covers every circuit of $M$.

In fact, we conjecture that this property holds in general.

**Conjecture 1.** *If $M=(S,\mathscr{B})$ is a matroid such that $S$ can be decomposed into two disjoint bases $A$ and $B$, then there exists an alternating path between $A$ and $B$ that covers every circuit of $M$.*

A seemingly weaker version would be the following.

**Conjecture 2.** *If $M=(S,\mathscr{B})$ is a matroid such that $S$ can be decomposed into two disjoint bases $A$ and $B$, then there exists an alternating cycle between $A$ and $B$ that covers every circuit of $M$.*

**Problem 6.** Prove that Conjectures 1 and 2 are equivalent.

A matroid $M$ is **base orderable** if for any two bases $A,B$ there exists a bijection $\varphi : A \to B$ such that

(BO) $$A - a + \varphi(a) \text{ and } B + a - \varphi(a) \text{ are bases for every } a \in A.$$

In other words, there exists a graph $G$ consisting of a matching between the elements of $A$ and $B$ such that $G$ covers the fundamental circuits of the elements of $A$ with respect to $B$, and the fundamental circuits of the elements of $B$ with respect to $A$.

Natural relexations of Conjectures 1 and 2 would be the following.

**Conjecture 3.** *If $M = (S,\mathscr{B})$ is a matroid such that $S$ can be decomposed into two disjoint bases $A$ and $B$, then there exists an alternating path/cycle between $A$ and $B$ that covers the fundamental circuits of the elements of $A$ with respect to $B$, and the fundamental circuits of the elements of $B$ with respect to $A$.*

**Problem 7.** Let $M$ be a paving matroid of rank at least 4. Prove that for any pair of disjoint bases $A$ and $B$ there exists a bijection $\varphi\colon A \to B$ satisfying (BO).

**Problem 8.** Prove that Conjecture 3 is true with a spanning tree in place of a path.

**Problem 9.** Prove that Conjecture 3 is true with a 2-factor in place of a cycle.