15/16th Emléktábla Workshop

Extremal combinatorics / Matroid theory 07.14 - 07.18.2024.

Preliminary Schedule for 16^{th} group:

Sunday

10:00 Arrival to the hotel
15:00-18:00 Short introduction/presentation of problems - Part I
18:00 Dinner
19:00 Short introduction/presentation of problems - Part II (if needed)

Monday – Wednesday

7:30-8:30 Breakfast
9:00 Grouping, work in groupd
12:30 Lunch
17:00 Presentation of daily progress
18:00 Dinner

Thursday

7:30-8:30 Breakfast
9:00
12:30 Lunch
15:00 Final presentations, farewell, check-out

List of Participants

Ahmad Abdi, *London School of Economics*
Kristóf Bérczi, *MTA-ELTE Matroid Optimization Research Group*
Erika Bérczi-Kovács, *Eötvös Loránd University*
Karthekeyan Chandrasekaran, *University of Urbana-Champaign*
Gergely Csáji, *Eötvös Loránd University*
Mahsa Dalirrooyfard, *London School of Economics*
Tamás Fleiner, *Budapest University of Technology and Economics*
András Frank, *Eötvös Loránd University*
Dániel Garamvölgyi, *Rényi Institute*
Panna Gehér, *Eötvös Loránd University*
Florian Hörsch, *CISPA Saarbrücken*
Áron Jánosik, *Eötvös Loránd University*
Tibor Jordán, *Eötvös Loránd University*
Naonori Kakimura, *Keio University*
Csaba Király, *HUN-REN-ELTE Egerváry Research Group*
Tamás Király, *Eötvös Loránd University*
Yusuke Kobayashi, *RIMS Kyoto*
Shubhang Kulkarni, *University of Urbana-Champaign*
Siyue Liu, *Carnegie Mellon University*
Vasilis Livanos, *University of Chile*
Péter Madarasi,*HUN-REN-ELTE Egerváry Research Group*
Bence Mátravölgyi, *ETH Zürich*
Jannik Matuschke, *KU Leuven*
Lydia Mirabel Mendoza-Cadena, *MTA-ELTE Matroid Optimization Research Group*
Ryuhei Mizutani, *University of Tokyo*
Taihei Oki, *University of Tokyo*
Neil Olver, *London School of Economics*
Gyula Pap, *Eötvös Loránd University*
Jeremy Quail, *University of Vermont*
Ildikó Schlotter, *Centre for Economic and Regional Studies*
Tamás Schwarcz, *MTA-ELTE Matroid Optimization Research Group*
András Sebő, *CNRS, Laboratoire G-SCOP, Univ. Grenoble*
Dániel Szabó, *Eötvös Loránd University*
Eszter Szabó, *Eötvös Loránd University*
Márton Szőke, *Budapest University of Technology and Economics*
Lilla Tóthmérész, *Eötvös Loránd University*
Kitti Varga, *HUN-REN-ELTE Egerváry Research Group*
Soma Villányi, *HUN-REN-ELTE Egerváry Research Group*
Chao Xu, *UESTC*
Yutaro Yamaguchi, *Osaka University*
Yu Yokoi, *Tokyo Institute of Technology*

Strongly connected re-orientations and polarity

by Ahmad Abdi

Let $D = (V, A)$ be a digraph with m arcs whose underlying undirected graph is 3-edge-connected. A *strongly connected re-orientation* is a subset $J \subseteq A$ such that $D \triangle J$ (the digraph obtained after flipping the orientations of the arcs in J) is strongly connected. Consider the set-system

$$\mathrm{SCR}(D) := \{\chi_J : J \text{ is a strongly connected re-orientation}\} \subseteq \{0,1\}^m.$$

This set-system enjoys several appealing discrete geometric properties. Let $S := \mathrm{SCR}(D)$ for short. For instance,

- S is *antipodally symmetric*, that is, a point belongs to S iff its antipodal point belongs to S: $p \in S$ iff $\mathbf{1} - p \in S$.

- S is *strictly connected*, that is, between every pair of points in S there is a monotone path on the skeleton graph of $\{0,1\}^m$ where all the intermediate nodes also belong to S. This follows from [3] and uses the 3-edge-connectivity of the underlying undirected graph of D.

  Let us describe this property in a different but equivalent way. Denote by G_m the skeleton graph of $\{0,1\}^m$. Then, if $S' \subseteq \{0,1\}^{m'}$ is a *restriction of* S (i.e. it is obtained from S after restricting some coordinates to 0 or 1 and then dropping the coordinates altogether), then the subgraph $G_{m'}[S']$ induced on S' is connected.

- Finally, S is *cube-ideal*, that is, its convex hull is described by hypercube and generalized set covering inequalities. More specifically, it is described by

$$\begin{aligned} x &\geq \mathbf{0} \\ x &\leq \mathbf{1} \\ \sum_{a \in \delta^+(U)} x_a + \sum_{a \in \delta^+(U)} (1 - x_a) &\geq 1 \quad \forall U \subsetneq V, U \neq \emptyset. \end{aligned}$$

  To see this, note first that the integer solutions to this system are precisely the points in S. Secondly, note that the generalized set covering inequalities form a submodular flow system, which in turn is box-TDI and so box-integral. This implies that the system above is integral.

The underpinning theme is to understand which restrictions of S have antipodal points.

Given disjoint $I, J \subseteq A$, consider the set-system obtained from $S \cap \{x : x_a = 0 \ \forall a \in I, x_b = 1 \ \forall b \in J\}$ after dropping the coordinates in $I \cup J$; we call this the *restriction of* S obtained after 0-restricting I and 1-restricting J.

Problem 1. *Find sufficient conditions on $I, J \subseteq A$ such that the restriction S' of S obtained after* 0*-restricting I and* 1*-restricting J, contains antipodal points.*

An obvious necessary condition is that the points in S' do not agree on a coordinate, that is, $S' \subseteq \{x : x_i = a\}$ for some $i \in [m']$ and $a \in \{0,1\}$.

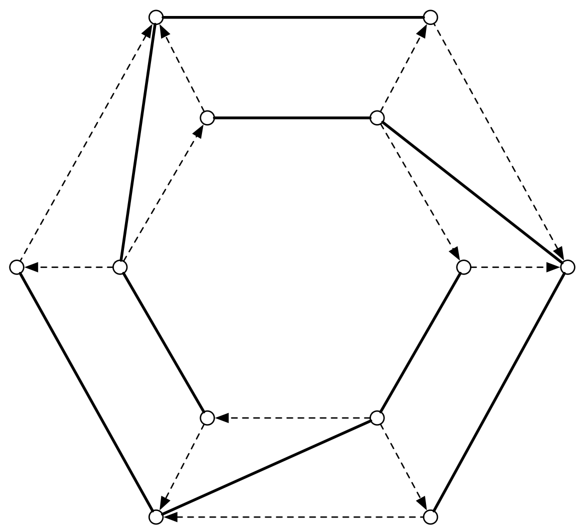
Let $S' \subseteq \{0,1\}^{m'}$ be a set-system. We say that S' is *polar* if either $S' \subseteq \{x : x_i = a\}$ for some $i \in [m']$ and $a \in \{0,1\}$, or S' contains antipodal points; otherwise it is *non-polar*.

For instance, the following conjecture has been made in relation to the problem above.

Conjecture 2. *[2] If $A - (I \cup J)$ is a spanning tree of D, then the restriction S' of S obtained after* 0*-restricting I and* 1*-restricting J, is polar.*

July 11, 2024 update: This conjecture has recently been solved in the affirmative, by Meike Neuwohner and myself. Together with Mahsa Dalirrooy-Fard, we are hoping to extend this result to the case when $A-(I\cup J)$ is a spanning forest with 2 connected components.

Not all restrictions of S are polar. For instance, consider the digraph D below, let I be the set of dashed arcs, and let $J:=\emptyset$. (The orientation of the solid arcs is irrelevant, you can orient them arbitrarily.) It can be shown that if S' is the restriction of S obtained after 0-restricting I, then the points in S' do not agree on a coordinate (every cut either has an incoming dashed arc or has at least two solid arcs), yet S' does not contain antipodal points (the solid arcs cannot be oriented in such a way such that every cut either has an incoming dashed arc, or has solid arcs crossing in both directions).



S' is *strictly non-polar* if it is non-polar and every restriction is polar. Observe that every non-polar set has a restriction that is strictly non-polar. My hope is that the discrete geometric properties above are helpful in tackling the above problems. More specifically,

Problem 3. *What can be said about strictly non-polar set-systems that are both cube-ideal and strictly connected?*

In [1], cube-ideal strictly non-polar sets were studied, and partly characterized, as they are helpful in generating *ideal minimally non-packing clutters* (for instance, they give rise to 716 such clutters with at most 14 elements). Interestingly, the notion of strict connectivity showed up and was studied there, albeit in a different context. I suspect there is more to be said.

References

[1] Abdi, A., Cornuejols, G., Guricanova, N., and Lee, D.: Cuboids, a class of clutters. Journal of Combinatorial Theory, Series B, 142:144–209, 2020.

[2] Chudnovsky, M., Edwards, K., Kim, R., Scott, A., and Seymour, P.: Disjoint dijoins. Journal of Combinatorial Theory, Series B, 120:18–35, 2016.

[3] Fukuda, K., Prodon, A., and Sakuma, T.: Notes on acyclic orientations and the shelling lemma. Theor. Comput. Sci., 263:9–16, 2001.

Uniform Covering by Common Bases

by Kristóf Bérczi

Let $M = (E, \mathcal{B})$ be a rank-r matroid whose ground set decomposes into two disjoint bases. Furthermore, assume that E is colored by r colors, each color appearing exactly twice. A basis of M is called *rainbow* if it does not contain two elements of the same color. The following problem was considered in [1].

Problem 1. *What is the minimum number of rainbow bases needed to cover E?*

By using matroid intersection, one can show that E can be covered by $\lfloor \log_2 |E| \rfloor + 1$ rainbow bases. On the other hand, the graphic matroid of K_4, where opposite pairs of edges form the color classes, shows that at least three such bases might be needed, and this value is believed to be the correct answer.

Now consider two matroids $M_1 = (E, \mathcal{B}_1)$ and $M_2 = (E, \mathcal{B}_2)$ on the same ground set, and assume that E decomposes into two bases in both of them. We propose the following conjecture.

Conjecture 2. *M_1 and M_2 has four common bases that covers each element exactly twice.*

Assuming that Conjecture 2 is true, the bound of 3 for Problem 1 follows by leaving out one of the four common bases – here, we think of the coloring as a partition matroid with color classes of size 2.

The problem is also related to the problem raised by Ahmad. Namely, let $G = (V, A, E)$ be a mixed graph where A and E denote the sets of directed and undirected edges, respectively. By a *dicut* of G, we mean a set $\emptyset \subsetneq Z \subsetneq V$ that has only incoming arcs in A. Suppose that for any dicut Z, the degree of Z in E is at least 2. This condition ensures that for any edge $e \in E$, if we orient e arbitrarily, then the remaining edges in $E - e$ can be oriented in such a way that the resulting digraph (including the arcs in A) is strongly connected.

Problem 3. *What is the minimum number of strongly connected orientations of G if every edge in E must be used in both directions?*

Since orientations of E that result in a strongly connected orientation of G are in one-to-one correspondence with the common bases of two matroids whose ground set decomposes into two disjoint bases, Conjecture 2 would imply a bound of 3 for this problem as well. From the two underlying matroids M_1 and M_2, M_2 is simply a partition matroid. However, the definition of M_1 is more involved and uses a positively intersecting supermodular function. The following question is then related to Ahmad's problem.

Problem 4. *Consider the setting of Problem 3 and assume that E forms a spanning tree/tree/spanning forest with 2 connected components. Is the matroid M_1 strongly base orderable in such cases?*

References

[1] Hörsch, F., Kaiser, T., Kriesell, M. (2024). Rainbow bases in matroids. SIAM Journal on Discrete Mathematics, 38(2), 1472-1491.

Cut Balanced Orientation

by Karthekeyan Chandrasekaran

Let $G = (V, E)$ be an undirected graph, $r \in V$, and k be an integer. An orientation $\overrightarrow{E}$ of E is *r-rooted k-cut-balanced* if $d^{out}_{\overrightarrow{E}}(U) \geq (1/k)d_E(U)$ for every $U \subseteq V$ such that $r \in U$. Using nowhere zero flows, we can show that every 2-edge-connected graph admits an r-rooted 6-cut-balanced orientation for every root vertex r.

Problem 1. *Does every 2-edge-connected graph admit an r-rooted 5-cut-balanced orientation for every root vertex r?*

One could also consider the $\{s,t\}$-separating version: Let $G = (V, E)$ be an undirected graph, $s, t \in V$, and k be an integer. An orientation $\overrightarrow{E}$ of E is *(s,t)-separating k-cut-balanced* if $d^{out}_{\overrightarrow{E}}(U) \geq (1/k)d_E(U)$ for every $s \in U \subseteq V - t$. Using nowhere zero flows, we can show that every 2-edge-connected graph admits an $\{s,t\}$-separating 6-cut-balanced orientation for every pair of vertices s, t .

Problem 2. *Does every 2-edge-connected graph admit an (s,t)-separating 5-cut-balanced orientation for every pair of vertices s, t?*

Finding near-feasible stable matchings in Resident-allocation

Gergely Csáji

Here, we are given a set of hospitals H, and a set of residents R, which can be partitioned into single residents S and couples C. A couple c_i consists of two residents (r_i, r_i').

A hospital h has a strict ranking $\succ_h$ over the acceptable residents, a single resident r_i has a strict ranking $\succ_{r_i}$ over the acceptable hospitals and each couple c_i has a strict ranking $\succ_{c_i}$ over acceptable pairs of hospitals.

Let M be a feasible matching.

- A pair (r, h), $r \in S, h \in H$ *blocks* M, if $h \succ_r M(r)$ and h has a free seat or there is a resident $r' \in M(h)$ such that $r \succ_h r'$.
- A couple $c_i = (r_i, r_i')$ *blocks* with a pair (h, h') of two distinct hospitals $h \neq h'$ if $(h, h') \succ_{c_i} M(c_i)$ and h has a free seat or a resident $r \in M(h)$ such that $r_i \succeq_h r$ and h' has a free seat or a resident $r' \in M(h)$ such that $r_i' \succeq_{h'} r'$ (i.e. it could happen that $r_i = r$ or $r_i' = r'$).
- A couple $c_i = (r_i, r_i')$ *blocks* with a hospital h, if $(h, h) \succ_{c_i} M(c_i)$ and both r_i and r_i' are among the best $q(h)$ residents in $M(h) \cup \{r_i, r_i'\}$.

We say that M is *stable*, if no such blocking coalition exists.

A stable matching may not exist and is NP-hard to find even in extremely restricted settings. However, we have a very promising recent result.

Theorem 1. *(Nguyen and Vohra) [2] Let I be an instance of the* HRC. *Then, there always exists $q'(h)$ capacities for each $h \in H$ satisfying that $|q(h) - q'(h)| \leq 2$ $\forall h \in H$, such that there is a stable integral matching M with respect to the $q'(h)$ capacities.*

Sadly, the way [2] finds such a near-feasible stable matching is by starting with a stable fractional solution, which is PPAD-hard to find [1].

Problem 1. *We have an existential result for a near-feasible stable matching. Can we find one in polynomial-time?*

References

[1] Gergely Csáji: *On the complexity of stable hypergraph matching, stable multicommodity flow and related problems* - Theoretical Computer Science

[2] Nguyen, Thanh and Vohra, Rakesh *Near-feasible stable matchings with couples* - American Economic Review,

Stable matchings in TU hypergraphs

Gergely Csáji

Let $\mathcal{H} = (V, E)$ be a hypergraph with capacities $q(v)$ and strict preferences $\succ_v$ for $v \in V$.

Given a feasible matching M (i.e. it respects the capacities), a hyperedge f is *blocking* M, if $f \notin M$ and for each $v \in f$, either v is unsaturated or there is an $f_v \in M$, such that $v \in f_v$ and $f \succ_v f_v$.

A matching M is called *stable*, if there isn't any blocking hyperedge.

The corresponding decision problem is the following.

SHM
Input: A hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $q(v) \in \mathbb{Z}$ capacities and $\succ_v$ strict preferences.
Question: Is there a stable hypergraph matching M?

In general, SHM is NP-hard, even in the 3-regular, 3-uniform case. [1]

There is a central related lemma of Scarf [2].

Lemma 1 (Scarf [2])**.** *Let Q be an $n \times m$ nonnegative matrix, such that every column of Q has a nonzero element and let $q \in \mathbf{R}^{\mathbf{n}}_{+}$. Suppose every row i has an strict ordering $>_i$ on those columns j for which $Q_{ij} > 0$. Then there is an extreme point of $\{Qx \le q,\ x \ge 0\}$, that dominates every column in some row, where we say that $x \ge 0$ dominates column j in row i, if $Q_i x = q_i$ and $j \le_i k$ for all $k \in \{1, \dots, m\}$, such that $Q_{ik} x_k > 0$.*

By Scarf's lemma, a stable hypergraph matching is guaranteed to exist, if the underlying hypergraph is TU.

An interesting special case: The University Dual Admission (UDA) problem is defined as follows. We have a set $U = \{u_1, \dots, u_n\}$ of universities, a set $C = \{c_1, \dots, c_k\}$ of companies and a set $S = \{s_1, \dots, s_m\}$ of students. For each university u_i, each company c_j may have a program p_{ij} at the university. Each university $u_i \in U$ has a capacity $c(u_i)$. Furhtermore, each program p_{ij} has a quota $q(p_{ij})$. Let the set of programs be denoted by P, while the set of programs at university u_i be denoted by P_i. We assume that the companies have no aggregate capacity over their programs, and we even allow the companies to have different ranking over the students for different programs. Hence, the programs can be treated independently. We reindex the programs in a way such that the programs at university u_i are indexed by $p_{i1}, p_{i2}, \dots, p_{ik_i}$, where k_i is the number of programs at the university.

The students may apply to only a university u_i or both a university u_i and a program p_{ij}, which is available at university u_i. By making a dummy company with dummy programs for each university that have large enough quotas, we can assume that each student applies to a university-program pair (u_i, p_{il}).

Hence, for each student s_j, we assume a strict preference list $\succ_{s_j}$ over the acceptable university-program pairs. Each university u_i has a strict preference order $\succ_{u_i}$ over the students and each program p_{il} has a $\succ_{p_{il}}$ strict ordering over the students.

We aim to find an assignment M of the students that is *feasible*, so $|M(s_j)| \le 1$, $|M(u_i)| \le c(u_i)$, $|M(p_{ij})| \le q(p_{ij})$.

We say that a feasible matching M is *stable*, if there is no (s_j, u_i, p_{il}) student-university-program triple, such that $(u_i, p_{il}) \succ_{s_j} M(s_j)$, u_i is unsaturated or there is a student $s_{j'} \in M(u_i)$ such that $s_j \succ_{u_i} s_{j'}$ or $s_j \in M(u_i)$ and the program p_{il} is either unsaturated or there is a student $s_{j''} \in M(p_{il})$ such that $s_j \succ_{p_{il}} s_{j''}$.

Theorem 2. *The hypergraph $\mathcal{H}$ on $S \cup U \cup P$ defined by the (mutually) acceptable (s_j, u_i, p_{il}) triples is a network hypergraph, meaning that its incidence matrix is a network matrix.*

Problem 1. *We know that because of this, a stable matching always exists in* UDA*. Can we find one in polynomial-time?*

References

[1] Gergely Csáji: *On the complexity of stable hypergraph matching, stable multicommodity flow and related problems* - Theoretical Computer Science

[2] Herbert Scarf: *The core of an N person game,* - Econometrica: Journal of the Econometric Society,

Simultaneous Min Cut

by Naonori KAKIMURA

The question is to determine the approximability of the problem below.

Problem 1. *We are given k undirected graphs $G_1, \ldots, G_k$ on the same vertex set V, and we denote by f_i the cut function of G_i. The problem is to find a vertex subset X such that $f_i(X)$ is at most d. In other words, we want to minimize the maximum of f_i's.*

It is known that the problem is FPT parameterized by $k + d$ [3], as the directed s-t cut variant (with terminals) is FPT. It would also be interesting to design a specific fixed-parameter algorithm for the problem using the property of the global cuts, which may lead to generalizing to symmetric submodular functions.

Related to the problem, the problem of maximizing the minimum of f_i's (simultaneous Max-Cut) is known to have a 0.8780-approximation algorithm [2], but to be inapproximable within a factor of $.87856 - 10^{-5}$ (under the UGC) [1]. It would also be interesting to fill in the gap as mentined in [4].

References

[1] Amey Bhangale, Subhash Khot, Simultaneous Max-Cut Is Harder to Approximate Than Max-Cut. CCC 2020: 9:1–9:15.

[2] Amey Bhangale, Subhash Khot, Swastik Kopparty, Sushant Sachdeva, and Devanathan Thiruvenkatachari. Near-optimal approximation algorithm for simultaneous max-cut. SODA, 1407–1425, 2018.

[3] Stefan Kratsch, Shaohua Li, Dániel Marx, Marcin Pilipczuk, and Magnus Wahlström. Multi-budgeted directed cuts. Algorithmica, 82(8):2135–2155, 2020.

[4] Mohit Singh and Santosh S. Vempala, Group Fairness in Optimization and Clustering, Optima Newsletter, 106, 2023.

Finding a Dicycle in a Basis Exchange Graph

by Yusuke Kobayashi

For a matroid $\mathcal{M} = (V, \mathcal{I})$ and for two disjoint bases B_1 and B_2 of $\mathcal{M}$, let $D_{\mathcal{M}}(B_1, B_2)$ denote the bipartite directed graph whose vertex set and arc set are $B_1 \cup B_2$ and

$$\{(u,v) \mid u \in B_1, v \in B_2, B_1 + v - u \in \mathcal{I}\} \cup \{(v,u) \mid u \in B_1, v \in B_2, B_2 + u - v \in \mathcal{I}\},$$

respectively.

Problem 1. *Suppose we are given a matroid $\mathcal{M} = (V, \mathcal{I})$ of rank r as an independence oracle and we are also given two disjoint bases B_1 and B_2 of $\mathcal{M}$. Can we construct a randomized algorithm for finding a dicycle in $D_{\mathcal{M}}(B_1, B_2)$ with high probability that uses $o(\sqrt{r})$ independene oracle queries, or can we prove that $\Omega(\sqrt{r})$ independence oracle queries are necessary?*

This problem appeared in a subroutine of an approximation algorithm for the submodular maximization problem under a matroid constraint. By the basis exchange property of matroids, every vertex is contained in a dicycle of length two, which can be found by using $O(r)$ independene oracle queries. Recently, Kobayashi and Terao [1] gave an algorithm for finding a dicycle in $D_{\mathcal{M}}(B_1, B_2)$ with high probability that uses $O(\sqrt{r}\,\mathrm{polylog}(r))$ independene oracle queries. I'm interested in whether this is best possible.

References

[1] Y. Kobayashi and T. Terao: Subquadratic Submodular Maximization with a General Matroid Constraint, ICALP 2024.

k-Distant Matroid

by Ryuhei Mizutani

For a finite set S and a positive integer k, a set function $f : 2^S \to \mathbf{R}$ is called *k-distant submodular* if the submodular inequality holds for every $X, Y \subseteq S$ with $|X \Delta Y| \geq k$, where $X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$. As a generalization of submodular function minimization, I recently showed that k-distant submodular functions can be minimized in polynomial time for a fixed positive integer k [1]. Considering that the rank function of a matroid is submodular, is it possible to consider a relaxation of matroids whose rank function is k-distant submodular?

Problem 1. *What is the axioms for an independence system or a basis family whose rank function is k-distant submodular?*

Problem 2. *Can we construct an efficient algorithm to obtain a maximum weight independent set of such an independence system?*

References

[1] R. Mizutnai, A polynomial algorithm for minimizing k-distant submodular functions, arXiv:2407.05127.

Proximity of Group-labeled Matroid Bases

by Taihei Oki

Recent research [1, 2] deals with matroids with group-label constraints, which are related to parity, congruency, and exact-weight constraints.

Let M be a matroid equipped with a labeling $\psi : E(M) \to \Gamma$ from the ground set $E(M)$ of M to an abelian group Γ. The *label* of $X \subseteq E(M)$ is defined to be $\psi(X) := \sum_{e \in X} \psi(e)$. For $F \subseteq \Gamma$, a basis B of M is called *F-avoiding* if $\psi(B) \notin F$.

Conjecture 1 ([1, Conjecture 5.1])**.** *For any basis B, there exists an F-avoiding basis B^* with $|B \setminus B^*| \leq |F|$, provided that at least one F-avoiding basis exists.*

Conjecture 1 is true if $|F| \leq 4$, Γ is an ordered group (e.g. $\Gamma = \mathbb{Z}$), or the matroid is strongly base orderable [1]. A relaxed conjecture that still leads us to a polynomial-time algorithm parameterized by $|F|$ can also be found in [1].

The following weighted variant is also interesting.

Conjecture 2 ([1, Conjecture 7.1])**.** *Let $w : E(M) \to \mathbb{R}$ be a weight function. For any minimum-weight basis B, there exists a minimum-weight F-avoiding basis B^* with $|B \setminus B^*| \leq |F|$, provided that at least one F-avoiding basis exists.*

Conjecture 2 is true for cases where $|F| = 1$ [1, 2] or the matroid is strongly base orderable [1].

An F-avoiding basis is called a *non-zero basis* if $F = \{0\}$ (0 can be changed to any group element). Non-Zero Matroid Intersection is the problem that, given two matroids $M_1 = (E, \mathcal{B}_1)$, $M_2 = (E, \mathcal{B}_2)$ and a group labeling $\psi : E \to \Gamma$, find a non-zero common basis, i.e., $B \in \mathcal{B}_1 \cap \mathcal{B}_2$ with $\psi(B) \neq 0$. There is a dichotomy theorem for non-zero matroid intersection.

Theorem 1 ([1, Theorems 3.7 and 6.1])**.** Non-Zero Matroid Intersection *can be solved in polynomial-time if and only if $\mathbb{Z}_2 \not\leq \Gamma$.*

Can we generalize this dichotomy theorem to the weighted case? F-avoiding matroid intersection with $|F| \geq 2$ is also open.

References

[1] F. Hörsch, A. Imolay, R. Mizutani, T. Oki, and T. Schwarcz. Problems on group-labeled matroid bases. In *Proc. of ICALP '24*, to appear.

[2] S. Liu and C. Xu. On the congruency-constrained matroid base. In *Proc. of IPCO '24*, to appear.

Integrality gap for the common intersection of multiple matroids

by Neil Olver

Let $M_1, M_2, \ldots, M_k$ be k matroids on the same groundset E. Let P_i be the independence polytope of matroid M_i. Consider the problem of, given a weight vector $w : E \to \mathbb{R}_{\geq 0}$, finding a maximum weight set that is independent in all k matroids. We can phrase this as an integer program:

$$\begin{array}{lll} \max & w^T x & \\ \text{s.t.} & x \in P_i & i = 1, 2, \ldots, k \\ & x \in \{0,1\}^E & \end{array}$$

Problem 1. *What is the integrality gap of the LP relaxation obtained by dropping the integrality constraints?*

The conjecture is that it is $k-1$. For $k=2$, this is of course true — this simply says that the intersection of two matroid polytopes is integral. For $k=3$, it also holds [1]; the proof is via an iterative rounding/relaxation approach, also with an invocation of matroid intersection. What about $k>3$?

A proof of this conjecture would likely yield a $(k-1)$-approximation for this problem. Using local search methods rather than LP-based methods, a $(k-1+\epsilon)$-approximation is known, for any $\epsilon > 0$ [2]. This has no implication for the integrality gap, however.

References

[1] A. Linhares, N. Olver, C. Swamy and R. Zenklusen. Approximate multi-matroid intersection via iterative refinement. Mathematical Programming **183**:397–418, 2020.

[2] J. Lee, M. Sviridenko and J. Vondrák. Submodular maximization over multiple matroids via generalized exchange properties. Mathematics of Operations Research, **35**(4):795–806, 2010.

Packing Problems in Matroids

by András Sebő

A *packing* problem for a hypergraph (V, E) and weight function $w : V \to \mathbb{N}$ is asking for a maximum multi-set of members of E so that each $v \in V$ is contained in at most $w(v)$ of them. Hypergraphs for which the optimal value of this problem is equal to that of its fractional relaxation are called *max-flow-min-cut* (MFMC) hypergraphs.

If A is the $0-1$ incidence matrix of an MFMC hypergraph then both the linear programs min $c^T x, Ax \geq 1, x \geq 0$ and max $y^T x, yA \leq c, y \geq 0$ have integer optimal vectors x and y. For the latter linear program this is just a rewriting of the definition of the MFMC property; it is also said that the linear program is *totally dual integral* (TDI). For the former linear program integrality is implied by the MFMC property (Edmonds and Giles).

The hypergraph (V, E) or the incidence matrix A have the *packing property* if both linear programs have integer (i.e. $0-1$) optimal vectors for all $0-1-\infty$ functions.

Problem 1. *[2] A hypergraph has the MFMC property if and only if it has the packing property*

A series of other conjectures and results concern the MFMC property and the packing property, and we have an expert of these at the workshop: Ahmad Abdi, (co)author of the most recent results in the subject. Another open problem concerning MFMC hypergraphs is the *Integer Caratheodory Problem* concerning them. A relation between the two problems may be fruitful.

For some hypergraph classes it has been an open problem since more than 40 years whether the MFMC property implies the *Integer Caratheodory property*, that is, *whether they have also an optimal solution containing at most $|V|$ different elements*: this was the "Integer Caratheodory Problem". Bill Cunningham [3] raised it first for independent sets *in matroids*, Cook Fonlupt Schrijver [4] studied it in general, for further references and proofs in special cases see the "Integer Caratheodory problem"in Schrijver's books (Linear and Integer programming and Combinatorial Optimization), and [6]. It has been refuted in general, but for independent sets, bases, of matroids Gijswijt and Regts [5] gave a positive answer. We state a related open problem in combinatorial terms:

An *r-arobrescence* is an aprborescence of a digraph rooted in a vertex r.

Problem 2. *Given a digraph, If w is the sum of r-arborescences, is it also a sum of r-arborescenses, where the different r-arborescences have a linearly independent set of incidence vectors ?*

Note that r-arborescences are common bases of two matroids. The complexity of packing problem for common bases of matroids in general had been a long-standing open problem until Bérczi and Schwarcz [1] has proved it to be NP-hard.

Problem 3. *Is there any relation between the complexity of solving packing problems at least for common bases of matroids and their MFMC or Integer Caratheodory property ? For instance, is the statement of the preceding problem true for common bases of strongly base orderable matroids ? (For the latter the packing problem can be solved in polynomial time.)*

Note that arborescences are usually not intersections of strongly base orderable matroids: the last problem is not necessariky more difficult than the preceding one.

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Spanning Tree with Perfect Matching

by Yutaro Yamaguchi

A tree is called *strongly balanced* if on one side of the bipartition of the vertex set, there exists a vertex r such that r is a leaf and all the other vertices have degree 2. This condition holds if and only if

- the tree has a perfect matching M (which is unique), and
- there exists a leaf r such that every vertex is reachable from r with an M-alternating path.

In a bipartite graph, the family of strongly balanced spanning trees can be clearly written as matroid intersection (graphic + partition), and hence the problem of finding a minimum-weight strongly balanced spanning tree is tractable. This fact was utilized to design a nontrivial approximation algorithm for some kind of connectivity augmentation problem [2]. It is also interesting that this problem is a common generalization of two fundamental special cases of the weighted matroid intersection problem: the weighted bipartite matching problem and the weighted arborescence problem.

Recently, it was shown that it is NP-hard to test whether a given subcubic planar graph contains a strongly balanced spanning tree or not [1]. Natural questions are as follows.

Problem 1. *Is there a nontrivial graph class of non-bipartite graphs for which it can be tested in polynomial time whether a given graph contains a strongly balanced spanning tree?*

Problem 2. *Is there any nontrivial sufficient condition for non-bipartite graphs to contain a strongly balanced spanning tree?*

Regarding the latter question, for example, it was considered in what graphs every spanning tree has a perfect matching [3].

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Exact, Correct Parity, and Bounded Correct Parity Matching

by Yutaro Yamaguchi

The *exact matching problem (EM)* is as follows: given a graph in which each edge is colored by red or blue, find a perfect matching with exactly k red edges. It is known that there exists a *randomized* polynomial-time algorithm [4] for EM, but any *deterministic* polynomial-time algorithm is not known for more than 40 years since the problem was stated.

Recently, El Maalouly, Steiner, and Wulf [1] showed that the parity-constrained relaxation, called the *correct parity matching problem (CPM)*, admits a deterministic polynomial-time algorithm: find a perfect matching with $k' \equiv k \pmod 2$ red edges. This result is based on a linear algebraic trick with the aid of Lovász' algorithm [3] for finding a basis of the linear subspace spanned by perfect matchings in a graph. It is elegant but heavily depends on the fact that we are only interested in the parity of the number of red edges, from which it seems difficult to obtain a promising idea to tackle EM. A natural question is as follows.

Problem 1. *Is there a "purely graphic" determinisitic polynomial-time algorithm for CPM?*

A natural "purely graphic" approach to CPM is as follows. We first find a perfect matching M. If the number of red edges in M has the same parity as k, we are done. Otherwise, it suffices to find an M-alternating cycle with an odd number of red edges (which is essentially equivalent to CPM). One natural way to do it is, for each fixed edge $e \in M$, to find an M-alternating cycle through e with odd number of red edges (if exists). Unfortunately, this problem has turned out NP-hard [5, 6]; thus, we have to consider another way.

In contrast, if we restricted ourselves to bipartite graphs, CPM can be solved by the above approach, since finding an M-alternating cycle with odd number of red edges reduces to finding a directed cycle with an odd number of red edges in the residual graph. This also works for the optimization version, called the *bounded correct parity matching problem (BCPM)* [1]: find a perfect matching with $k' \equiv k \pmod 2$ red edges minimizing k'. BCPM is also a relaxation of EM, and hence it admits a randomized polynomial-time algorithm. A natural question is as follows.

Problem 2. *Is there a determinisitic polynomial-time algorithm for BCPM in non-bipartite graphs?*

These problems in bipartite graphs are naturally extended to matroid intersection, including other special cases such as arborescences. As a corollary of the results of the last workshop [2], it has been obtained that the correct parity common basis problem is difficult (requires an exponential number of oracle calls) even for the intersection of a partition matroid and a sparse paving matroid.

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Popularity in Matroid Intersection

by Yu Yokoi

In a manner analogous to extending the concept of stable matchings to matroid intersection [1, 2], the concept of popular matchings has been extended to matroid intersection by Kamiyama [3], who solved the maximum popular common independent set problem for weakly base orderable matroids. This tractability result was recently extended to general matroids by Csáji–Király–Yokoi [4]. Below are related problems left open by them. While the original problems are about popular common independent sets, here we present their common base variants for simplicity.

Voting in Matroid Intersection Let $M = (S, \mathcal{I}, \succ)$ be an ordered matroid, i.e., $(S, \mathcal{I})$ is a matroid and $\succ$ is a total order on the ground set S. We denote the base family by $\mathcal{B}$. Given an ordered pair of bases $(I, J) \in \mathcal{B} \times \mathcal{B}$, consider a bipartite graph $(I \setminus J, J \setminus I; E_{IJ})$, where

$$E_{IJ} = \{\, uv : u \in I \setminus J,\ v \in J \setminus I,\ I - u + v \in \mathcal{B} \,\}.$$

For two bases I and J and a perfect matching $N \subseteq E_{I,J}$ in this bipartite graph, we define

$$\mathsf{vote}(I, J, N) = |\{\, uv \in N : u \succ v \,\}| - |\{\, uv \in N : u \prec v \,\}|$$

where $u \in I \setminus J$ and $v \in J \setminus I$. Considering the most adversarial perfect matching for I, we define

$$\mathsf{vote}(I, J) = \min\{\, \mathsf{vote}(I, J, N) : N \text{ is a perfect matching in } E_{IJ} \,\}.$$

By considering the most favorable perfect matching for I (i.e., using “max” instead of “min” in the above definition), we can similarly define $\mathsf{max\text{-}vote}(I, J)$.

Popularity in Matroid Intersection Let $M_1 = (S, \mathcal{I}_1, \succ_1)$ and $M_2 = (S, \mathcal{I}_2, \succ_2)$ be ordered matroids with base families $\mathcal{B}_1$ and $\mathcal{B}_2$. For an ordered pair (I, J) of common bases and $i \in \{1, 2\}$, we define $\mathsf{vote}_i(I, J)$ as above with respect to M_i. For a common base $I \in \mathcal{B}_1 \cap \mathcal{B}_2$, we say that

- I is *popular* if $\mathsf{vote}_1(I, J) + \mathsf{vote}_2(I, J) \geq 0$ for every $J \in \mathcal{B}_1 \cap \mathcal{B}_2$,

While we know that a popular common base can be computed efficiently, we do not know the complexity of testing the popularity of a given common base.

Problem 1. *Given $I \in \mathcal{B}_1 \cap \mathcal{B}_2$, can we check whether I is popular or not efficiently?*

We remark that this problem is tractable if M_1 and M_2 are partition matroids as shown in [5], but its proof technique seems difficult to extend to the matroidal setting.

We next consider some variants of popularity. For a common base $I \in \mathcal{B}_1 \cap \mathcal{B}_2$, we say that

- I is *defendable* if $\mathsf{vote}_1(J, I) + \mathsf{vote}_2(J, I) \leq 0$ for every $J \in \mathcal{B}_1 \cap \mathcal{B}_2$.
- I is *weakly popular* if $\mathsf{max\text{-}vote}_1(I, J) + \mathsf{max\text{-}vote}_2(I, J) \geq 0$ for every $J \in \mathcal{B}_1 \cap \mathcal{B}_2$.

In a special case where M_1 and M_2 are partition matroids, defendability and weak popularity are equivalent. This follows from the fact that in this case the exchangeability graph for I and that for J are symmetric (i.e., $I - u + v \in \mathcal{B} \Leftrightarrow J + u - v \in \mathcal{B}$ for any $u \in I \setminus J$ and $v \in J \setminus I$) and hence $-\mathsf{vote}_i(J, I) = \mathsf{max\text{-}vote}_i(I, J)$ holds. However, for general matroids, there is no such symmetricity and hence the relation between the values of $\mathsf{vote}_i(I, J)$ (or $\mathsf{max\text{-}vote}_i(I, J)$) and $\mathsf{vote}_i(J, I)$ is unclear from the definitions.

In [4], the authors proved that $\mathsf{vote}_i(I, J) + \mathsf{vote}_i(J, I) \leq 0$ holds for any bases $I, J \in \mathcal{B}_i$, from which it follows that popularity implies defendability. Also, there is an example demonstrating that weak popularity does not imply defendability (that example consists of the graphic matroid of K_4 and a partition matroid $U_{1,2} \oplus U_{1,2} \oplus U_{1,2}$). So we know the following relations.

$$\begin{array}{ccc} & I \text{ is popular} & \\ & \Downarrow \quad\quad \Downarrow & \\ I \text{ is defendable} & \not\Leftarrow & I \text{ is weakly popular} \end{array}$$

We still don't know whether defendability implies weak popularity.

Problem 2. *Does defendability imply weak popularity?*

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