Matroid theory
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Exercise 1 (Kundu and Lawler)**.** Let $M_1 = (S, r_1)$ and $M_2 = (S, r_2)$ be matroids with closure operators σ_1 and σ_2, respectively. Let F_1 and F_2 be two common independent sets. Prove that there exists a common independent set F such that $F_1 \subseteq \sigma_1(F)$ and $F_2 \subseteq \sigma_2(F)$.

Exercise 2. Develop a min-max theorem for the maximum weight of a common independent set of i elements.

Exercise 3. Let $c^{(j)}$ denote the maximal c-weight of a j-element common independent set. Prove that there exists a j such that $c^{(0)} \leq \cdots \leq c^{(j)} \geq c^{(j+1)} \geq \cdots \geq c^{(k)}$, where k is the maximum size of a common independent set.

Exercise 4. Let M_1, M_2 be matroids over the same ground set S, and let $S_1, S_2 \subseteq S$. Give an algorithm that decides if there exist bases $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ such that $B_1 - B_2 \subseteq S_1$, $B_2 - B_1 \subseteq S_2$.

Exercise 5. Let M_1, M_2 be matroids over the same ground set S, and assume that S decomposes into two disjoint bases in both of them. Furthermore, let $c\colon S \to \mathbb{R}$ be a weight function. Prove that there exists a common basis of weight at least $c(S)/2$.

Exercise 6. Let M be a matroid over S. Given a vector $v \in \mathbb{Q}^S$, decide if v is contained in the independence polytope of M.

Exercise 7. Let $G = (V, E)$ be a graph. Give an algorithm that determines the minimum number of colors needed to color the edges in such a way that every cycle is rainbow colored.

Exercise 8. Given a finite set of points in a two-dimensional plane, find the maximum number of disjoint triples each of which defines a non-degenerate triangle.