Matroid theory
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Exercise 1 (Brualdi)**.** Let $G = (S, T; E)$ be a bipartite graph, and $M_1 = (S, r_1)$ and $M_2 = (T, r_2)$ be matroids. We call a matching $F \subseteq E$ strongly independent if it covers independent sets both in M_1 and M_2. Prove that the maximum size of a strongly independent matching is equal to

$$\min\{r_1(X) + r_2(Y) \mid X \subseteq S, Y \subseteq T, X \cup Y \text{ covers every edge of } G\}.$$

Exercise 2 (Kundu and Lawler)**.** Let $M_1 = (S, r_1)$ and $M_2 = (S, r_2)$ be matroids with closure operators σ_1 and σ_2, respectively. Let F_1 and F_2 be two common independent sets. Prove that there exists a common independent set F such that $F_1 \subseteq \sigma_1(F)$ and $F_2 \subseteq \sigma_2(F)$.

Exercise 3. Develop a min-max theorem for the maximum weight of a common independent set of i elements.

Exercise 4 (Krogdahl)**.** Prove that $c^{(k+1)} - c^{(k)} \leq c^{(k)} - c^{(k-1)}$, where $c^{(j)}$ denotes the maximal c-weight of a j-element common independent set.

Exercise 5. Let $G = (V, E)$ be a graph and $c_1, ..., c_q : E \to \mathbb{R}$ be q cost functions defined on its edges. Give an algorithm that finds q pairwise edge-disjoint spanning trees $T_1, \ldots, T_q$ minimizing $\sum_{i=1}^{q} c_i(T_i)$.

Exercise 6. Let $M = (S, r)$ be a loop-free matroid and $J \subseteq S$ be a subset of at most k elements. Prove the following.

(A) If S can be partitioned into k independent sets, then S can be partitioned into k independent sets in such a way that each of them contains at most one element of J.

(B) If there are k disjoint bases, then there are k disjoint bases in such a way that each of them contains at most one element of J and the union of them includes J.

Exercise 7. Let $M_1, \ldots, M_k$ be matroids over the same ground set S, and let I_i be an independent set of M_i for $i = 1, \ldots, k$. Prove that these sets can be extended to disjoint independent sets covering S if and only if

$$\sum_{i=1}^{k} [r_i(X \cup I_i) - |I_i|] \geq |X|$$

for all $X \subseteq S'$, where $S' = S - \cup_{i=1}^{k} I_i$.

Exercise 8. Lt $G = (V, E)$ be an undirected graph and $c\colon \binom{V}{2} \to \mathbb{R}_+$ be a cost function. Give an algorithm that determines a minimum cost graph $H = (V, F)$ for which $G + H = (V, E \cup F)$ contains k pairwise disjoint spanning trees.