Matroid theory
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From earlier weeks:

Exercise 1. As a further extension of the generalized submodular inequality, prove that $\hat{b}(c_1) + \hat{b}(c_2) \geq \hat{b}(c_1 + c_2)$ holds.

Exercise 2. Prove Rota's conjecture for strongly base orderable matroids.

Exercise 3. Prove Rota's conjecture for graphic matroids when each B_i is a star.

Exercise 4. Let $M = (S, r)$ be a matroid and $c : S \to \mathbb{Z}_+$ be a weight function. For $X \subseteq S$, let $b_c(X)$ denote the maximum weight of an independent subset of X. Prove that b_c is submodular.

Exercise 5 (For this, matroid intersection is needed, so we will discuss it later.)**.** Let $G = (V, E)$ be a graph and $c_1, ..., c_q : E \to \mathbb{R}$ be q cost functions defined on its edges. Give an algorithm that finds q pairwise edge-disjoint spanning trees $T_1, \ldots, T_q$ minimizing $\sum_{i=1}^{q} c_i(T_i)$.

New set of exercises:

Exercise 6. Let $M = (S, r)$ be a matroid. For a set $Z \subseteq S$, let $\mathrm{cl}(Z)$ denote the closure of Z, that is, $\mathrm{cl}(Z) = \{s \in S \mid r(Z + s) = r(Z)\}$. Verify the following:

(S1) $A \subseteq \mathrm{cl}(A)$,

(S2) $A \subseteq B \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$,

(S3) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$,

(S4) if $y \notin \mathrm{cl}(A)$ and $y \in \mathrm{cl}(A + x)$, then $x \in \mathrm{cl}(A + y)$.

Exercise 7. Let $\sigma\colon 2^S \to 2^S$ be a function satisfying (S1)–(S4). Prove that there exists a unique matroid M such that the closure operator of M is exactly σ.

Exercise 8. Let S and T be disjoint sets, M_1 be a matroid on S, and M_2 be a matroid on $S \cup T$ in which S is a basis. Let $M = (M_1 + M_2)/S$. Prove that $F \subseteq T$ is independent in M if and only if there exists $F_1 \in M_1$ such that $S - F_1 + F$ is a basis of M_2.

Exercise 9. Prove that the rank function of M in the previous exercise is $r(Z) = \min_{X \subseteq S}\{r_1(X) + r_2(Z \cup X) - |X|\}$.

Exercise 10. Let $b\colon 2^S \to \mathbb{Z}$ be a monotone increasing intersecting submodular function satisfying $b(\emptyset) = 0$. Prove that $b^\vee$ is also monotone increasing.

Exercise 11. Let $M = (S, \mathcal{F})$ be a laminar matroid. Find an intersecting submodular function $b\colon 2^S \to \mathbb{Z}$ such that $\mathcal{F} = \{I \subseteq S \mid |I \cap X| \leq b(X) \text{ for all } X \subseteq S\}$.

Exercise 12. Let $G = (V, E)$ be an undirected graph. Prove that G does not contain two trees (having at least one edge) spanning the same subset of vertices if and only if the rigidity matroid of G is the free matroid.

Exercise 13. Let $M_1 = (S, r_1)$ and $M_2 = (S, r_2)$ be two matroids on the same ground set. Show that the problem of finding a common basis of the two matroids can be reduced to the case when one of the matroids is a partition matroid with upper bound one on every partition class.