Matroid theory
Date: 18 March 2024

From earlier weeks:

Exercise 1. As a further extension of the generalized submodular inequality, prove that $\hat{b}(c_1) + \hat{b}(c_2) \geq \hat{b}(c_1 + c_2)$ holds.

Exercise 2. Prove Rota's conjecture for strongly base orderable matroids.

Exercise 3. Prove Rota's conjecture for graphic matroids when each B_i is a star.

Exercise 4. Let $M = (S, r)$ be a matroid and $c : S \to \mathbb{Z}_+$ be a weight function. For $X \subseteq S$, let $b_c(X)$ denote the maximum weight of an independent subset of X. Prove that b_c is submodular.

Exercise 5. Prove that transversal matroids are exactly the homomorphic images of partition matroids.

New set of exercises:

Exercise 6. Let $M = (S, \mathcal{F})$ be a matroid and k, ℓ be non-negative integers such that $k \geq \ell$. Consider the problem of paritioning S into k parts such that the union of any ℓ of them forms an independent set of M. Give characterizations for the existence of such a partition when $\ell = 1$, $k - 1$ and k.

Exercise 7. Let $G = (V, E)$ be a graph and $c_1, ..., c_q : E \to \mathbb{R}$ be q cost functions defined on its edges. Give an algorithm that finds q pairwise edge-disjoint spanning trees $T_1, \ldots, T_q$ minimizing $\sum_{i=1}^{q} c_i(T_i)$.

Exercise 8. Let A and B be bases of the matroid $M = (S, r)$. Prove that for any partition $A = A_1 \cup \cdots \cup A_q$ there exists a partition $B = B_1 \cup \cdots \cup B_q$ such that $A - A_i + B_i$ is a basis for $i = 1, \ldots, q$.

Exercise 9. Let A and B be bases of the matroid $M = (S, r)$. Prove that for any partition $A = A_1 \cup A_2$ there exists a partition $B = B_1 \cup B_2$ such that $A - A_i + B_i$ and $B - B_i + A_i$ are both bases for $i = 1, 2$.

Exercise 10. Assume that the ground set of the matroid M decomposes into two disjoint bases. Show that there are then an exponential number of such decompositions.

Exercise 11. Let $M = (S, \mathcal{B})$ be a matroid and $S_1, \ldots, S_q \subseteq S$. Give an algorithm for deciding if there exists a basis $B \in \mathcal{B}$ such that $B \cap S_i$ spans S_i for $i = 1, \ldots, q$.

Exercise 12. Let M be a matroid. Prove that the truncation and elongation operations are dual to each other in the sense that the dual of a truncation of M forms an elongation of M^*. Similarly, the dual of an elongation of M forms a truncation of M^*.

Exercise 13. Let $D = (V, A)$ be an acyclic directed graph. Let $\mathcal{F} = \{F \subseteq A \mid d_F^{in}(v) \leq 1 \text{ for every } v \in V\}$. Prove that $\mathcal{F}$ satisfies the independence axioms. What are the bases of the matroid thus obtained?