Matroid theory
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From earlier weeks:

Exercise 1. As a further extension of the generalized submodular inequality, prove that $\hat{b}(c_1) + \hat{b}(c_2) \geq \hat{b}(c_1 + c_2)$ holds.

Exercise 2. Prove that a paving matroid is sparse paving if and only if it has a hypergraph representation in which each hyperedge has size r.

Exercise 3. Let S be a ground set of size at least r, $\mathcal{H} = \{H_1, \dots, H_q\}$ be a (possibly empty) collection of subsets of S, and $r, r_1, \dots, r_q$ be non-negative integers satisfying

$$|H_i \cap H_j| \leq r_i + r_j - r \text{ for } 1 \leq i < j \leq q. \tag{H1}$$

(a) Prove that $\mathcal{I} = \{X \subseteq S \mid |X| \leq r,\ |X \cap H_i| \leq r_i \text{ for } 1 \leq i \leq q\}$ forms the independent sets of a matroid.

(b) Prove that the rank function of the matroid is $r_M(Z) = \min\left\{r, |Z|, \min\limits_{1 \leq i \leq q} \{|Z - H_i| + r_i\}\right\}$.

(c) Show that if

$$|S - H_i| + r_i \geq r \text{ for } i = 1, \dots, q \tag{H2}$$

holds, then the rank of the matroid is r.

(d) Prove that the hypergraph in Exercise 3 can be chosen in such a way that

$$r_i \leq r - 1 \text{ for } i = 1, \dots, q, \tag{H3}$$
$$|H_i| \geq r_i + 1 \text{ for } i = 1, \dots, q. \tag{H4}$$

Exercise 4. Verify the following.

(a) The class of elementary split matroids is closed under duality.

(b) The class of elementary split matroids is closed under taking minors.

(c) The class of elementary split matroids is closed under truncation.

Exercise 5. Let M be a rank-r elementary split matroid with a non-redundant representation $\mathcal{H} = \{H_1, \dots, H_q\}$ and $r, r_1, \dots, r_q$. Let F be a set of size r.

(a) If F is H_i-tight for some index i then F is a basis of M.

(b) If F is both H_i-tight and H_j-tight for distinct indices i and j then $H_i \cap H_j \subseteq F \subseteq H_i \cup H_j$.

New set of exercises:

Exercise 6. Let $G = (V, E)$ be a graph on n vertices. Prove that if E is colored with exactly $n - 1$ colors, then G either contains a rainbow cycle or a monochromatic cut.

Exercise 7. Prove Rota's conjecture for strongly base orderable matroids.

Exercise 8. Prove Rota's conjecture for graphic matroids when each B_i is a star.

Exercise 9. Prove that the graphic matroid of the complete graph K_4 on four vertices is not a transversal matroid.

Exercise 10. Using the Gallai-Edmonds decomposition of graphs, prove that every matching matroid is a transversal matroid.

Exercise 11. Let $M = (S, r)$ be a matroid and $c : S \to \mathbb{Z}_+$ be a weight function. For $X \subseteq S$, let $b_c(X)$ denote the maximum weight of an independent subset of X. Prove that b_c is submodular.

Exercise 12. Prove that transversal matroids are exactly the homomorphic images of partition matroids.