Matroid theory
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From previous weeks:

Exercise 1. Prove that the circuits of a matroid M are exactly the cuts of its dual M^*.

Exercise 2. A matroid M is called binary if it can be represented over the field $GF(2)$, that is, there exists a 0-1 matrix A whose columns are identified with the elements of the matorid in such a way that a subset of columns of A is independent over $GF(2)$ if and only if the corresponding elements form an independent set of M. Verify that the graphic matroid of any graph is a binary matroid.

Exercise 3. Prove that a matroid $M = (S, r)$ is connected if and only if its dual $M^* = (S, r^*)$ is connected.

Exercise 4. A matroid M is **strongly base orderable** if for any two bases A, B there exists a bijection $\varphi : A \to B$ such that

$$A - X + \varphi(X) \text{ is a basis for every } X \subseteq A. \quad \text{(SBO)}$$

Let A and B be disjoint spanning trees of the same simple undirected graph G. Prove that there is no bijection between A and B satisfying (SBO).

Exercise 5. Let $G = (V, E)$ be an undirected graph with $|V| = n$ such that E can be decomposed into two disjoint spanning trees A and B. Prove that there exists a bijection $\varphi : A \cup B \to \{1, \dots, 2n - 2\}$ for which every cycle of G contains two consecutive numbers.

New set of exercises:

Exercise 1. As a further extension of the generalized submodular inequality, prove that $\hat{b}(c_1) + \hat{b}(c_2) \geq \hat{b}(c_1 + c_2)$ holds.

A rank-r matroid $M = (S, \mathcal{I})$ is called a **paving matroid** if each circuit has size at least r, or in other words, each set of size at most $r - 1$ is independent. The matroid is **sparse paving** if M^* is also paving. For a non-negative integer r, a ground set S of size at least r, and a (possibly empty) family $\mathcal{H} = \{H_1, \dots, H_q\}$ of proper subsets of S such that $|H_i \cap H_j| \leq r - 2$ for $1 \leq i < j \leq q$, the set system $\mathcal{B}_{\mathcal{H}} = \{X \subseteq S \mid |X| = r,\ X \not\subseteq H_i \text{ for } i = 1, \dots, q\}$ forms the set of bases of a paving matroid, and in fact every paving matroid can be obtained in this form. We will refer to this as a **hypergraph representation** of M.

Exercise 2. Give an example showing that the dual of a paving matroid is not necessarily paving.

Exercise 3. Prove that a paving matroid is sparse paving if and only if it has a hypergraph representation in which each hyperedge has size r.

Exercise 4. Let S be a ground set of size at least r, $\mathcal{H} = \{H_1, \dots, H_q\}$ be a (possibly empty) collection of subsets of S, and $r, r_1, \dots, r_q$ be non-negative integers satisfying

$$|H_i \cap H_j| \leq r_i + r_j - r \text{ for } 1 \leq i < j \leq q. \quad \text{(H1)}$$

(a) Prove that $\mathcal{I} = \{X \subseteq S \mid |X| \leq r,\ |X \cap H_i| \leq r_i \text{ for } 1 \leq i \leq q\}$ forms the independent sets of a matroid.

(b) Prove that the rank function of the matroid is $r_M(Z) = \min\left\{r, |Z|, \min_{1 \leq i \leq q} \{|Z - H_i| + r_i\}\right\}$.

(c) Show that if
$$|S - H_i| + r_i \geq r \text{ for } i = 1, \dots, q \tag{H2}$$
holds, then the rank of the matroid is r.

(d) Prove that the hypergraph in Exercise 4 can be chosen in such a way that
$$r_i \leq r - 1 \text{ for } i = 1, \dots, q, \tag{H3}$$
$$|H_i| \geq r_i + 1 \text{ for } i = 1, \dots, q. \tag{H4}$$

Matroids that can be obtained as described in Problem 4 are called **elementary split matroids**. A matroid is a **split matroid** if it is a direct sum of a single elementary split matroid and some uniform matroids. We call the representation **non-redundant** if all of (H1)–(H4) hold. A set $F \subseteq S$ is called **H_i-tight** if $|F \cap H_i| = r_i$.

Exercise 5. Verify the following.

(a) The class of elementary split matroids is closed under duality.

(b) The class of elementary split matroids is closed under taking minors.

(c) The class of elementary split matroids is closed under truncation.

Exercise 6. Let M be a rank-r elementary split matroid with a non-redundant representation $\mathcal{H} = \{H_1, \dots, H_q\}$ and $r, r_1, \dots, r_q$. Let F be a set of size r.

(a) If F is H_i-tight for some index i then F is a basis of M.

(b) If F is both H_i-tight and H_j-tight for distinct indices i and j then $H_i \cap H_j \subseteq F \subseteq H_i \cup H_j$.