Continuous Optimization
Date: 23 November 2023
Submission deadline:
7 December, 12:00

Let $E(x_0, M)$ denote the ellipsoid $\{x \in \mathbb{R}^n \mid (x - x_0)^T M^{-1}(x - x_0) \leq 1\}$.

Exercise 1 (1pt)**.** Consider an affine map $\phi(x) = Ax + b$, where $A \in \mathbb{R}^{n\times n}$ is an invertible matrix. Show that $\phi(E(x_0, M)) = E(Ax_0 + b, AMA^T)$.

Exercise 2 (2pts)**.** Let $E(x_0, M) \subseteq \mathbb{R}^2$ be an ellipsoid where

$$x_0 = (1, 2) \text{ and } M = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}.$$

Let $\phi(x) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x + (3, 5)$. Show that M is positive definite. Write up the ellipsoid $\phi(E(x_0, M))$.

Exercise 3 (2pts)**.** Let both B_1 be the unit ball with the origin as a center and B_2 be the unit ball with the point $(1, 0)$ as a center in $\mathbb{R}^2$. Let

$$x_1 = (1, 1),\ x_2 = (2, 2),\ M_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } M_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Prove that there exists no affine mapping ϕ such that $\phi(B_1) = E(x_1, M_1)$ and $\phi(B_2) = E(x_2, M_2)$.

Exercise 4 (2pts)**.** Let $C = [0, 1] \times [0, 1] \subseteq \mathbb{R}^2$ and let $a = (3, 3)$. Find all hyperplanes that separates C and a.

Exercise 5 (2pts)**.** Consider the polytope

$$P = \{x \in \mathbb{R}^3 \mid x_1 + 3x_2 + x_3 \geq 3, x_1 - x_2 + x_3 \geq 5, x_2 + x_3 \leq 4, 0 \leq x_1 \leq 2, 0 \leq x_2 \leq 2\}.$$

Let $x_0 = (0, 0, 3)$ and $M_0 = 9I$, where I is the 3×3 identity matrix. Consider the ellipsoid $E_0 = E(x_0, M_0)$. Show that $P \subseteq E_0$.