Continuous Optimization
Date: 16 November 2023
Submission deadline:
23 November, 12:00

Exercise 1 (1pt)**.** Let $P = \{x \in \mathbb{R}^n \mid \langle a_i, x\rangle \leq b_i \text{ for } i = 1, \dots, m\}$ and $x \in int(P)$. For a vector $c \in \mathbb{R}^n$, let $c_x = H(x)^{-1}c$, where $H(x)$ is the Hessian of the logarithmic barrier function. Verify that the point $x - c_x/\|c_x\|_x$ is in P.

Exercise 2 (2pts)**.** Let $G = (V, E)$ be an undirected graph. The spanning tree polytope of G is the convex hull of the characteristic vectors of spanning trees of G. Define $P(G) = \{x \in \mathbb{R}^E_+ \mid x(\delta(S)) \geq 1 \text{ for } \emptyset \subset S \subset V\}$, where $\delta(S)$ denotes the set of edges having exaactly one endpoint in S.

(a) Give an example showing that P is not necessarily the spanning tree polytope of G.

(b) Show that even if we add the constraint $x(E) = |V| - 1$ to the above cut constraints, it still does not determine the spanning tree polytope.

Exercise 3 (2pts)**.** The perfect matching polytope of a graph G is given by $P_{PM}(G) = \{x \in \mathbb{R}^E_+ \mid x(\delta(v)) = 1 \text{ for } v \in V,\ x(\delta(U)) \geq 1 \text{ for } U \subseteq V, |U| \text{ odd}\}$. Separating over $P_{PM}(G)$ means that for a vector $x \in \mathbb{R}^E$, we want to decide if $x \in P_{PM}(G)$ or not.

(a) Prove that separating over $P_{PM}(G)$ reduces to the following **odd minimum cut problem:** given $x \in \mathbb{Q}^E$, find
$$\min_{S \subseteq V, |S| \text{ odd}} \sum_{u \in S, v \notin S, uv \in E} x_{uv}.$$

(b) Prove that the odd minimum cut problem is polynomially solvable.

Exercise 4 (1pt)**.** Let $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a polyhedron and t be a vector with all coordinates equal to $\alpha > 0$. Show that $\{x \in \mathbb{R}^n \mid Ax \leq b + t\}$ is full dimensional.