Continuous Optimization
Date: 9 November 2023
Submission deadline:
16 November, 12:00

Exercise 1 (1pt)**.** Explain how to find a steepest descent direction in the ℓ_∞-norm, and give a simple interpretation.

Exercise 2 (1pt)**.** Determine the square root of 2, accurate to six decimal places, using Newton's method (your argument for accuracy should not rely on comparison with a result obtained by a calculator).

Exercise 3 (1pts)**.** Let $P = \{x \in \mathbb{R}^2 : |x_1| \leq 1, |x_2| \leq 1\}$ and $c = (1, 1) \in \mathbb{R}^2$. Characterize the points of the central path Γ_c.

Exercise 4 (2pts)**.** Intuitively, if the gradient of the logarithmic barrier at a given point x is short, then the point x is far away from the boundary of the polytope. In particular, the analytic center lies well inside the polytope. More formally, consider the following problem. Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polytope. Suppose that $x_0 \in P$ is a point such that $b_i - \langle a_i, x_0\rangle \geq \delta$ for every $i = 1, \ldots, m$ and some $\delta > 0$. Prove that if D is the diameter of P (in the Euclidean norm) then for every $x \in P$ we have $b_i - \langle a_i, x\rangle \geq \delta \cdot (m + \|g(x)\| \cdot D)^{-1}$ for $i = 1, \ldots, m$, where $g(x)$ is the gradient of the logarithmic barrier function at x.

Exercise 5 (3pts)**.** Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polytope. For $x \in P$, let $H(x)$ denote the Hessian of the logarithmic barrier function for P. We define the Dinkin ellipsoid at a point $x \in P$ as $E_x = \{y \in \mathbb{R}^n : (y - x)^T H(x)(y - x) \leq 1\}$. Let x_0^* be the analytic center.

(a) Prove that for all $x \in \mathrm{int}(P)$, $E_x \subseteq P$.

(b) Assume without loss of generality (by shifting the polytope) that $x_0^* = 0$. Prove that $P \subseteq mE_{x_0^*}$.

(c) Prove that if the set of constraints is symmetric, i.e., for every constraint of the form $\langle a', x\rangle \leq b'$ there is a corresponding constraint $\langle a', x\rangle \geq -b'$, then $x_0^* = 0$ and $P \subseteq \sqrt{m}E_{x_0^*}$.