Continuous Optimization
Date: 28 September 2023
Submission deadline:
5 October, 12:00

Exercise 1 (2pts)**.** Let us consider the following functions:

$$f_1(w_1, w_2) = \frac{1}{2}w_1^2 + \frac{7}{2}w_2^2$$
$$f_2(w_1, w_2) = 100(w_2 - w_1^2)^2 + (1 - w_1)^2 \qquad (Rosenbrock's function)$$
$$f_3(w_1, w_2) = \frac{1}{2}w_1^2 + w_1 \cos w_2.$$

(a) Calculate the gradient of the functions.

(b) Determine the global minimum of the functions.

(c) Are these function convex?

Exercise 2 (2pts)**.** Let $G = (V, E)$ be an undirected graph and let B be the $|E| \times |V|$ matrix such that the ith row corresponds to the ith edge. Let the ith edge be uv. Then the ith row of B is $\chi_u - \chi_v$, where χ_u is the vector in $\mathbb{R}^{|V|}$ with one in the uth position and zero otherwise. The $|V| \times |V|$ matrix L defined by $B^T B$ is called the Laplacian matrix of the graph.

(a) Prove that $L = D - A$, where D is the diagonal degree matrix (i.e. $D_{v,v} = deg(v)$), and A is the adjacency matrix of the graph (i.e. $A_{u,v} = 1$ if $uv \in E$).

(b) Show that the all-1 vector is in the kernel of L.

(c) Prove that when the graph is connected, the kernel is just the one-dimensional space spanned by the all-1 vector.

Exercise 3 (1pt)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentialbe function. Prove that if $\|\nabla f(x)\| \leq G$ for all $x \in \mathbb{R}^n$ and some $G > 0$, then f is G-Lipschitz, that is, $|f(x) - f(y)| \leq G\|x - y\|$ for every $x, y \in \mathbb{R}^n$.

Exercise 4 (1pt)**.** Let $G = (V, E)$ be an undirected graph and $s, t \in V$. Consider the following problem:

$$\begin{aligned} \min \sum_{uv \in E} |x_u - x_v| \\ \text{s.t.} \qquad x_s - x_t = 1 \end{aligned}$$

This is not a linear program in this form. Rewrite it as a linear program.

Exercise 5 (1pt)**.** Given a convex, differentiable function $F : K \to \mathbb{R}$ over a convex subset K of $\mathbb{R}^n$, the *Bregman divergence* of $x, y \in K$ is defined as $D_F(x, y) = F(x) - F(y) - \langle \nabla F(y), x - y \rangle$. Prove that $D_F(x, y) \geq 0$.

Exercise 6 (1pt)**.** Define a function F for which $D_F(x, y) = \| x - y \|_2^2$.

Exercise 7 (1pt)**.** Let $F : K \to \mathbb{R}$ be a convex, differentiable function, and let $x, y, z \in K$. Prove that

$$\langle \nabla F(y) - \nabla F(z), y - x \rangle = D_F(x, y) + D_F(y, z) - D_F(x, z).$$