Continuous Optimization
Date: 21 September 2023
Submission deadline:
28 September, 12:00

Exercise 1 (1pt). Verify that the set of positive semidefinite matrices forms a convex cone.

Exercise 2 (1pt). Show that $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if $g(t) := f(ty + (1-t)x)$ is convex for any $x, y \in dom(f)$.

Exercise 3 (2pts). Prove that

(a) e^{ax} is convex on $\mathbb{R}$ for any $a \in \mathbb{R}$,

(b) x^a is convex on $\mathbb{R}_{>0}$ when $a \geq 1$ or $a \leq 0$, and is concave when $0 < a < 1$,

(c) $\log(x)$ is concave on $\mathbb{R}_{>0}$,

(d) $x \log(x)$ is convex on $\mathbb{R}_{>0}$.

Exercise 4 (2pts). Show the following.

(a) (Nonnegative weighted sum) If $w_i \geq 0$ and f_i is convex for every i, then $f = \sum_i w_i f_i$ is convex.

(b) (Affine mapping) If f is convex, then $g(x) = f(Ax + b)$ is convex.

(c) (Pointwise maximum) If f_i is convex for every i, then $f(x) = \max_i\{f_i(x)\}$ is convex.

(d) (Composition) If g is convex and h is convex and non-decreasing, then $f(x) = h(g(x))$ is convex.

Exercise 5 (1pt). Let S_1 an S_2 be convex sets. Prove that $S_1 - S - 2 = \{x - y | x \in S_1, y \in S_2\}$ is also convex.

Exercise 6 (1pt). Prove that for an arbitrary set $S \subseteq \mathbb{R}^n$, the polar set

$$S^* = \{y \in \mathbb{R}^n \mid y^T x \leq 1 \ \forall x \in S\}$$

is convex.

Exercise 7 (1pt). Prove that for an arbitrary function $f : \mathbb{R}^n \to \mathbb{R}$, the conjugate function

$$f^* = \sup\{y^T x - f(x) \mid x \in dom(f)\}$$

is convex.

Exercise 8 (2pts). Consider the optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}} \quad & x^2 + 2x + 4 \\ s.t. \quad & x^2 - 4x \leq -3 \end{aligned}$$

(a) Solve this problem, i.e., find the optimal solution.

(b) Is this a convex program?

(c) Derive the dual problem $\max_{\lambda \geq 0} g(\lambda)$. Find g and its domain.

(d) Prove that weak duality holds.

(e) Is Slater's condition satisfied? Does strong duality hold?