Continuous Optimization
Date: 14 September 2023
Submission deadline:
21 September, 12:00

Exercise 1 (1pt)**.** Given a matrix $A \in \mathbb{R}^{n\times n}$, then λ is an eigenvalue of A if and only if $det(A - \lambda I) = 0$.

Exercise 2 (1pt)**.** Given an $n \times n$ PD matrix H and a vector $a \in \mathbb{R}^n$, prove that $aa^T \preceq H$ of and only if $1 \geq a^T H^{-1} a$.

Exercise 3 (1pt)**.** Given a symmetric matrix A, if λ_1, u_1 and λ_2, u_2 are two eigenvalue-eigenvector pairs such that $\lambda_1 \neq \lambda_2$ then $\langle u_1, u_2 \rangle = 0$.

Exercise 4 (2pts)**.** Let $M \in \mathbb{R}^{n\times n}$ be a real symmetric matrix. Prove that the following characterizations are equivalent:

(i) all eigenvalues of M are non-negative,

(ii) $x^T M x \geq 0$ for all $x \in \mathbb{R}^n$,

(iii) $M = V^T V$ for some $V \in \mathbb{R}^{n\times n}$.

Exercise 5 (1pt)**.** Prove that if M is an $n \times n$ real symmetric matric that is PD, then $M = BB^T$ for some $n \times n$ real matrix B with linearly independent rows.

Exercise 6 (1pt)**.** Prove that all the eigenvalues of an $n \times n$ real symmetric matrix are real. Show that every eigenvalue of a PSD matrix is nonnegative.

Exercise 7 (2pts)**.** Verify that the triangle inequalities are satisfied for the following norms:

(a) ℓ_∞-norm, $\| x \|_\infty = \max\{|x_1|, \ldots, |x_n|\}$,

(b) quadratic norm, $\| x \|_M = \sqrt{x^T M x} = \| M^{\frac{1}{2}} x \|_2$ for a positive definite $M \in \mathbb{R}^{n\times n}$.

Exercise 8 (1pt)**.** Show that for an $n \times n$ real symmetric matrix A with eigenvalues $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$, its matrix norm is $\|A\|_2 = \max\{|\lambda_1(A)|, |\lambda_n(A)|\}$. If in addition A is PSD, then $\|A\|_2 = \lambda_n(A)$.

Exercise 9 (1pt)**.** Let $\| \cdot \|$ be a norm. Verify that the dual norm $\|x\|_* = \sup_{y \in \mathbb{R}^n : \|y\| \leq 1} \langle x, y \rangle$ is indeed a norm.

Exercise 10 (1pt)**.** Let $p, q \in \mathbb{R}_+$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Prove that the norms $\| \cdot \|_p$ and $\| \cdot \|_q$ are dual to each other.