Optimization

Fall semester 2022/23

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Outline

Question 1

Let $G = (V, E)$ be an undirected graph and $s, t \in V$. Consider the following problem:

$$\min \sum_{uv \in E} |x_u - x_v| \tag{1}$$

$$\text{s.t. } x_s - x_t = 1 \tag{2}$$

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Then we have the LP:

$$\min \sum_{uv \in E} y_{uv} \tag{3}$$

$$\text{s.t. } x_s - x_t = 1 \tag{4}$$

$$x_u - x_v \leqslant y_{uv} \quad \forall uv \in E \tag{5}$$

$$-x_u + x_v \leqslant y_{uv} \quad \forall uv \in E \tag{6}$$

$$y_{uv} \geqslant 0 \quad \forall uv \in E \tag{7}$$

Question 2, f_1

$f_1(w_1, w_2) = \frac{1}{2}w_1^2 + \frac{7}{2}w_2^2$

a. Calculate the gradients of the functions. (2pts)
b. Are these function convex? (2pts)
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Obs. $D = 5$, thus for a given ϵ, we will only require $T = \mathcal{O}(\frac{7\times 5^2}{\epsilon})$ steps to get a point close to the minimizer $(0,0)$.

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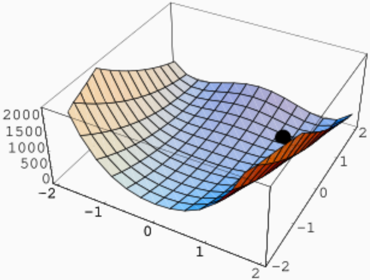

(b)

https://mathworld.wolfram.com/RosenbrockFunction.html

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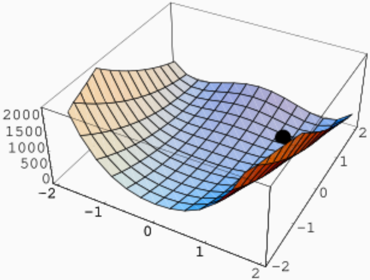

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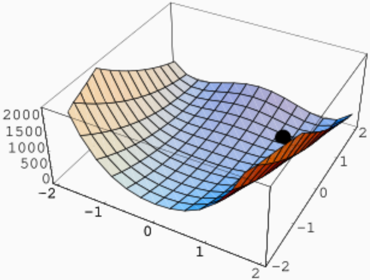

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d. $L = 400 \Rightarrow \eta = 1/400$

$x_1 = (2, 2)$

$x_2 = (2, 2) - \frac{1}{400}(1602, -400) = (-2.005, 3)$

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Given a convex, differentiable function $F : K \to \mathbb{R}$ over a convex subset K of $\mathbb{R}^n$, the Bergman divergence of $x, y \in K$ is defined as

$$D_F(x, y) = F(x) - F(y) - \langle \nabla F(y), x - y \rangle$$

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