Optimization

Fall semester 2022/23

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Outline

Set 3

Question 1

Is it true, that a set $K \subseteq \mathbb{R}^n$ is convex if and only if for any $x, y \in K$ we have $(x + y)/2 \in K$? (1pt)

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False.

Consider $K = (0, 1) \cap \mathbb{Q}$.

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False.

Consider $K = (0, 1) \cap \mathbb{Q}$.

It holds that $\forall x, y \in K$ we have $(x + y)/2 \in K$ but K is clearly not convex.

Question 2

Prove that for an arbitrary function $f : \mathbb{R}^n \to \mathbb{R}$, the conjugate function $f^*(y) = \sup\{y^T x - f(x) | x \in \operatorname{dom} f\}$ is convex. (1pt)

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Let $h(y) := y^T x - f(x) \quad$ for $x \in \operatorname{dom} f$.

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Let $\theta \in [0, 1]$, and $a, b \in \mathbb{R}^n$

$$h(\theta a + (1 - \theta) b) = (\theta a + (1 - \theta) b)^T x - f(x) \tag{1}$$

$$\tag{4}$$

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$$= \theta a^T x + (1-\theta) b^T x - f(x) \pm \theta f(x) \tag{2}$$

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Now we take the supremum on both sides,

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that is, $f^*(\theta a + (1-\theta) b) = \theta f^*(a) + (1-\theta) f^*(b)$

Question 3, a-c

Verify the following statements. (5pts)

- (a) e^{ax} is convex on $\mathbb{R}$ for any $a \in \mathbb{R}$.
- (b) x^a is convex on $\mathbb{R}_{>0}$ when $a \geqslant 1$ or $a \leqslant 0$, otherwise it is concave.
- (c) $\log x$ is concave on $\mathbb{R}_{>0}$.
- (d) $x \log x$ is convex on $\mathbb{R}_{>0}$.
- (e) $\max\{x_1, \ldots, x_n\}$ is convex on $\mathbb{R}^n$

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Recall. Second order condition for convexity: Let f be twice differentiable such that $\operatorname{dom} f$ is open. Then f is convex $\iff \nabla^2 f(x) \geqslant 0 \ \forall x \in \operatorname{dom} f$

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Recall. Second order condition for convexity: Let f be twice differentiable such that $\mathrm{dom}\, f$ is open. Then f is convex $\iff \nabla^2 f(x) \geqslant 0 \ \forall x \in \mathrm{dom}\, f$

(a). $f'(x) = ae^{ax} \implies f''(x) = a^2 e^{ax} \geqslant 0$.

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(a). $f'(x) = ae^{ax} \implies f''(x) = a^2 e^{ax} \geqslant 0.$

(b). $f'(x) = ax^{a-1} \implies f''(x) = a(a-1)x^{a-2} \begin{cases} \geqslant 0 & \text{if } a \geqslant 1 \\ < 0 & \text{if } 0 < a < 1 \\ \geqslant 0 & \text{if } a \leqslant 0 \end{cases}.$

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(c). $f'(x) = \frac{1}{x} \implies f''(x) = -\frac{1}{x^2} \leqslant 0.$

Question 3, d-e

Verify the following statements. (5pts)

- (d) $x \log x$ is convex on $\mathbb{R}_{>0}$.
- (e) $\max\{x_1, \ldots, x_n\}$ is convex on $\mathbb{R}^n$

Question 3, d-e

Verify the following statements. (5pts)

- **d** $x \log x$ is convex on $\mathbb{R}_{>0}$.
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(d). $f'(x) = \log x + 1 \implies f''(x) = \frac{1}{x} \geqslant 0$.

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(e). Let $h(x) = \max\{x_1, \ldots, x_n\}$. Let $x, y \in \mathbb{R}^n$, $\theta \in [0, 1]$.

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(e). Let $h(x) = \max\{x_1, \ldots, x_n\}$. Let $x, y \in \mathbb{R}^n$, $\theta \in [0, 1]$.
For any $i = 1, \ldots, n$ it holds

$$\theta x_i + (1 - \theta) y_i \leqslant \theta \max_{j \in \{1,\ldots,n\}} \{x_j\} + (1 - \theta) \max_{j \in \{1,\ldots,n\}} \{y_j\}.$$

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Then,

$$\max_{i \in \{1,\ldots,n\}} \theta x_i + (1-\theta) y_i \leqslant \theta \max_{j \in \{1,\ldots,n\}} \{x_j\} + (1-\theta) \max_{j \in \{1,\ldots,n\}} \{y_j\}$$

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Then,

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$$\implies h\left(\theta x + (1-\theta) y\right) \leqslant \theta h(x) + (1-\theta) h(y)$$

Question 4, a

Consider the optimization problem

$$\min\ x^2 + 1$$
$$\text{s.t. } (x-2)(x-4) \leqslant 0$$
$$x \in \mathbb{R}$$

Analysis of primal problem. Give the feasible set, the optimal value, and the optimal solution. (1pt)

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The feasible set is $2 \leqslant x \leqslant 4$.

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The feasible set is $2 \leqslant x \leqslant 4$.

The optimal solution is $x^* = 2$.

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The feasible set is $2 \leqslant x \leqslant 4$.

The optimal solution is $x^* = 2$.

The optimal value is 5.

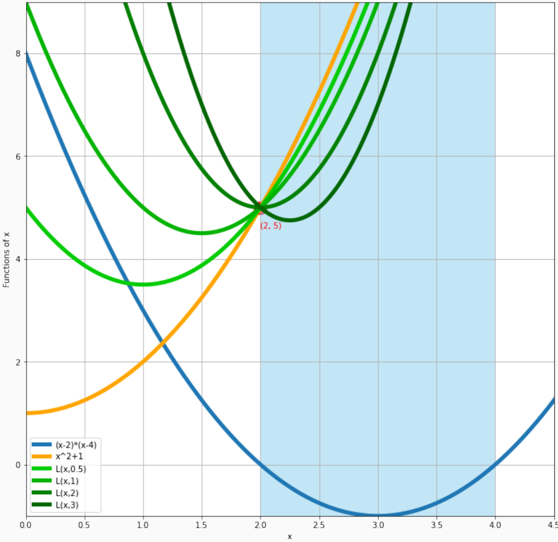
Question 4, b, plot

Consider the optimization problem

$$\min\ x^2 + 1$$
$$\text{s.t. } (x-2)(x-4) \leqslant 0$$
$$x \in \mathbb{R}$$

Lagrangian and dual function. Plot the objective $x^2 + 1$ versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x,\lambda)$ versus x for a few positive values of λ. Verify the lower bound property, that is, $y^* \geqslant \inf_x L(x,\lambda)$ for $\lambda \leqslant 0$. Derive and sketch the Lagrange dual function g. (2pts).

$$L(x,\lambda) = x^2 + 1 + \lambda(x-2)(x-4)$$



Question 4, b, Lagrange dual function

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$$L(x, \lambda) = (1+\lambda)x^2 - 6\lambda x + 8\lambda + 1$$

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$$L(x, \lambda) = (1+\lambda)x^2 - 6\lambda x + 8\lambda + 1$$

We obtain $\inf_x L(x, \lambda)$:

Question 4, b, Lagrange dual function

Consider the optimization problem

$$\min \; x^2 + 1$$
$$\text{s.t. } (x-2)(x-4) \leqslant 0$$
$$x \in \mathbb{R}$$

Lagrangian and dual function. Plot the objective x^2+1 versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x,\lambda)$ versus x for a few positive values of λ. Verify the lower bound property, that is, $y^* \geqslant \inf_x L(x,\lambda)$ for $\lambda \leqslant 0$. Derive and sketch the Lagrange dual function g. (2pts).

$L(x,\lambda) = (1+\lambda)x^2 - 6\lambda x + 8\lambda + 1$

We obtain $\inf_x L(x,\lambda)$:

$\frac{\partial}{\partial x} L(x,\lambda) = 2(1+\lambda)x - 6\lambda = 0$

Question 4, b, Lagrange dual function

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Lagrangian and dual function. Plot the objective $x^2 + 1$ versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property, that is, $y^* \geqslant \inf_x L(x, \lambda)$ for $\lambda \leqslant 0$. Derive and sketch the Lagrange dual function g. (2pts).

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$\Rightarrow x = \frac{3\lambda}{(1+\lambda)}$ is the minimizer for $L(x, \lambda)$

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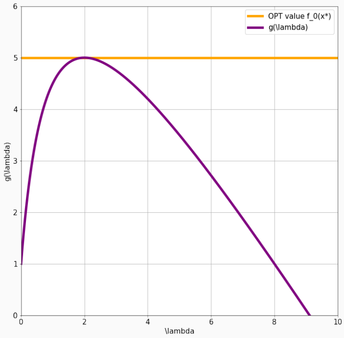
Then, $g(\lambda) = \inf_x L(x, \lambda) = -\frac{9\lambda^2}{1+\lambda} + 8\lambda + 1$



Question 4, c

Consider the optimization problem

$$\min \; x^2 + 1$$
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Lagrange dual problem.

State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution λ^*. Does strong duality hold? (2pts)

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The Lagrange dual problem:

$$\max\ -\frac{9\lambda^2}{1+\lambda} + 8\lambda + 1$$
$$\text{s.t. } \lambda \geqslant 0$$

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We find its optimal solution:

$$g'(\lambda)=\frac{9\lambda^2}{(1+\lambda)^2}-\frac{18\lambda}{(1+\lambda)}=0$$

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We find its optimal solution:

$$g'(\lambda) = \frac{9\lambda^2}{(1+\lambda)^2} - \frac{18\lambda}{(1+\lambda)} = 0$$

$$\iff 9\lambda^2 - 18\lambda(1+\lambda) + 8(1+\lambda)^2 = 0$$

Question 4, c

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$$\iff 9\lambda^2 - 18\lambda(1+\lambda) + 8(1+\lambda)^2 = 0$$
$$\iff (\lambda - 2)(\lambda + 4) = 0$$

Optimal Lagrangian dual solution: $\lambda^* = 2$.

Optimal Lagrangian dual value:

$g(2) = -\frac{9(2)^2}{1+2} + 8(2) + 1 = 5$