Optimization

Fall semester 2022/23

Kristóf Bérczi,
Lydia Mendoza

Eötvös Loránd University
Institute of Mathematics
Department of Operations Research

Outline

Set 2

Question 1, a

Consider the following integer programming problem.

$$\max \; x_1 + 2x_2 \quad (1)$$

$$\text{s.t.} \; -3x_1 + 4x_2 \leqslant 4 \quad (2)$$

$$3x_1 + 2x_2 \leqslant 11 \quad (3)$$

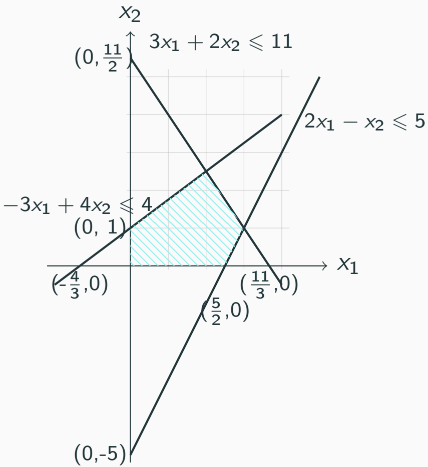
$$2x_1 - x_2 \leqslant 5 \quad (4)$$

$$x_1, x_2 \geqslant 0 \quad (5)$$

$$x_1, x_2 \text{ integer} \quad (6)$$

Use a figure to answer the following questions.

(a)What is the optimal cost of the linear programming relaxation? What is the optimal cost of the integer programming problem? (1pt)



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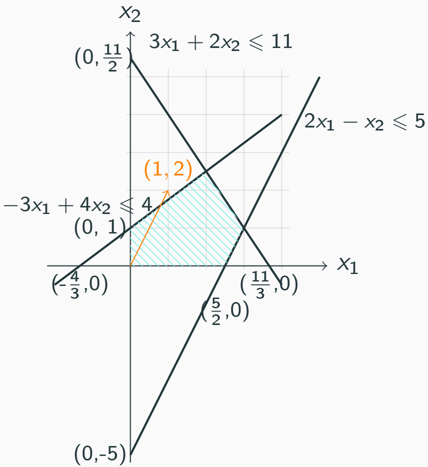

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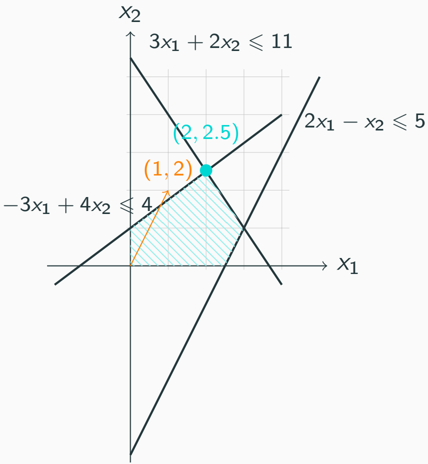

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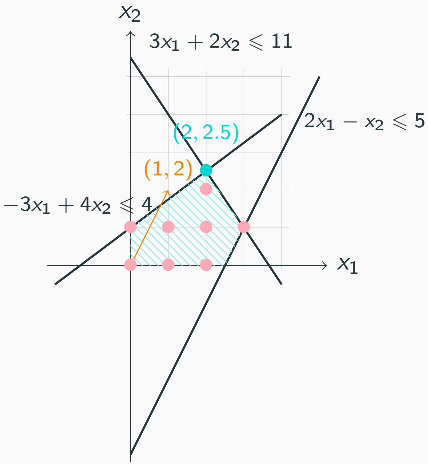

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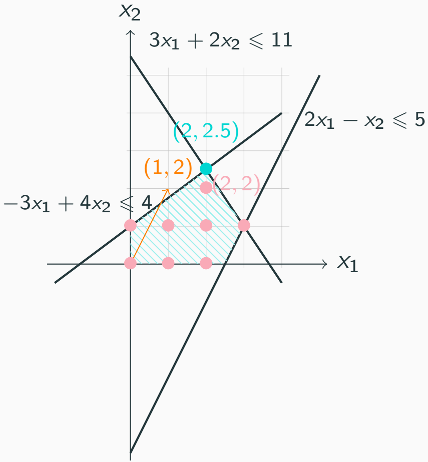

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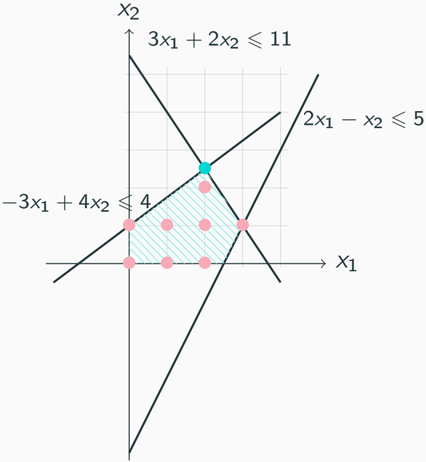
Use a figure to answer the following questions.

(b) What is the convex hull of the set of all solutions to the integer programming problem? (1pt)



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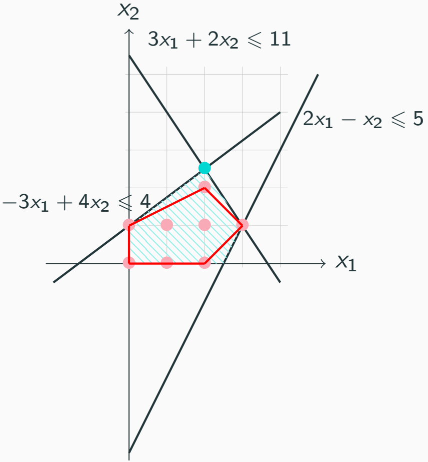

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A company is manufacturing k different products using m resources. The amounts of available resources are given, together with the requirement of each of them for the different products. The selling price of the products are also known.

(a) Write up an IP model that aims at maximizing the total profit. (1pt)

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$$\max \sum_{i=1}^{k} c_i x_i \tag{13}$$

$$\text{s.t.} \sum_{i=1}^{k} a_{ij} x_i \leqslant b_j \quad \forall j = 1, 2, \ldots, m \tag{14}$$

$$x_i \text{ integer} \qquad \forall i = 1, 2, \ldots, k \tag{15}$$

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$$\max \sum_{i=1}^{k} c_i x_i - s_i y_i \tag{16}$$

$$\text{s.t.} \sum_{i=1}^{k} a_{ij} x_i \leqslant b_j \quad \forall j = 1, 2, \ldots, m \tag{17}$$

$$x_i \leqslant M y_i \quad \forall i = 1, 2, \ldots, k \tag{18}$$

$$x_i \text{ integer} \quad \forall i = 1, 2, \ldots, k \tag{19}$$

$$y_i \in \{0, 1\} \quad \forall i = 1, 2, \ldots, k \tag{20}$$

Question 3

Consider the integer programming problem

$$\begin{aligned} \min \ & x_{n+1} \\ \text{s.t. } & 2x_1 + 2x_2 + \cdots + 2x_n + x_{n+1} = n \\ & x_i \in \{0, 1\} \end{aligned}$$

Show that any branch and bound algorithm that uses LP relaxations to compute lower bounds, and branches by setting a fractional variable to either zero or one, will require the enumeration of an exponential number of subproblems when n is odd. (2pts)

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When n is even, we can set

$$x_i = \begin{cases} 1 & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases}.$$

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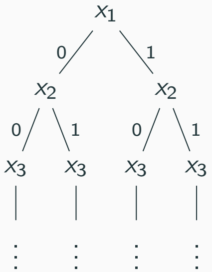
We construct the tree:



1 node $= 2^0$

2 nodes $= 2^1$

4 nodes $= 2^2$

$\vdots$

Question 4

The pagination problem faced by a document processing program like LaTeXcan be abstracted as follows. The text consists of a sequence $1, \ldots, n$ of n items (words, formulas, etc.). A page that starts with item i and ends with item j is assigned an attractiveness factor c_{ij}. Assuming that the factors c_{ij} are available, we wish to maximize the total attractiveness of the paginated text. Develop an algorithm for this problem. (Hint: try to use recursive approach.) (2pts)

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$A[i, j+1] = \max\{A[i, j-1], c_{j-1,j}\}$